274 Curves on Surfaces, Lecture 13

Dylan Thurston

Fall 2012

14 More about laminations

Recall that last time we were trying to understand the Thurston compactification of (decorated) Teichmüller space. When we did this we got points whose coordinates satisfied the tropical Ptolemy relations. We can imagine doing this for other kinds of Teichmüller space:

Teichmüller space	Coordinates	Integer limits	Coordinates
$\tilde{\mathcal{T}}_{g,n}$	λ-lengths	Integer laminations	Intersections
$\mathcal{T}_g$	Lengths of closed curves	Integer laminations	Intersections
$\mathcal{T}_{g,0,n}$	Shear coordinates	?	?

Here $\mathcal{T}_{g,0,n}$ denotes the Teichmüller space of surfaces with no punctures but with geodesic boundary, and we should take logarithms of shear coordinates before projectivizing because they behave multiplicatively. We do not yet know what the corresponding limit points of the Thurston compactification are.

Recall that one geometric interpretation of the cross-ratio was in terms of the distance (shear) between two midpoints obtained by dropping angle bisectors: if d is this distance, then $\tau = e^d$. Recall also that we obtained limit points by inserting a simple multicurve and inserting necks; then in the limit, lengths become proportional to intersection numbers.

Accordingly, we should count intersections of the segment between two midpoints; these are also called shear coordinates. We will count these with sign: in one direction they will be assigned $+1$ while in another direction they will be assigned -1.

Alternately, for any choice of decoration we had

$$\tau(E) = \frac{\lambda(B)\lambda(D)}{\lambda(A)\lambda(C)}. \tag{1}$$

Tropicalizing this relation gives almost, but not quite, the right answer: since $\lambda(A) = e^{\frac{\ell}{2}}$, we should take

$$\text{shear}(E) = \frac{\ell(B) + \ell(D) - \ell(A) - \ell(C)}{2}. \tag{2}$$

If we don't want to use decorations, we should work with infinitely many curves since ideal polygons have sides of infinite length. This gives in the limit a kind of metric space where the distance between two points is given by the intersection number of a geodesic between them. We can identify points at distance zero, and then a triangle with infinitely many curves at the vertices (since an ideal triangle has infinite length) becomes an infinite tripod tree. Gluing two such triangles together to obtain a quadrilateral gives us a metric tree.

This gives limit points described by multicurves such that there are infinitely many curves around punctures, some of which may be spiraling (to account for geodesic boundary). These are *unbounded laminations.*

Exercise 14.1. *How do these shear coordinates transform under change of triangulation? Compare to the geometric answer (from points in noncompactified Teichmüller space).*

When we draw a geodesic representative of a complicated multicurve on, say, the 4-punctured sphere, it does not look much like a curve because it becomes very close to itself. What we get looks more like a train track. As the curve becomes more complicated (under the action of the mapping class group) we get something which should describe a limit point in the compactification, which corresponds to a train track with real weights. In this particular case we should replace Fibonacci numbers with the corresponding powers of the golden ratio.

Exercise 14.2. *Apply the mapping class group element we have been applying to the limit train track. Check that you get the same thing up to scale and splitting.*

(It is probably a better idea to apply the inverse, which will make the train track simpler rather than more complicated.)

This reflects the fact that the Thurston compactification is a ball, so any element of the mapping class group has a fixed point. In general, studying the action of the mapping class group on the Thurston compactification led Thurston to the following theorem.

Theorem 14.3. *Every element ϕ of the mapping class group is in one of the following three categories:*

1. *ϕ has finite order.*

2. *ϕ is* reducible*: it fixes a finite collection of closed curves (e.g. Dehn twist) (possibly permuting them).*

3. *ϕ is* pseudo-Anosov*: it fixes exactly two points in the Thurston compactification, neither of which are points in Teichmüller space. Call these two projective measured laminations L_+, L_-. The lamination L_+ is, in an appropriate sense, $\lim_{n\to\infty} \phi^n(c)$ for any nontrivial multicurve c, and L_- is, in an appropriate sense, $\lim_{n\to\infty} \phi^{-n}(c)$.*

This is analogous to the classification of hyperbolic isometries into elliptic, parabolic, and hyperbolic elements. For example, ϕ finite order turns out to be equivalent to the claim that ϕ fixes a hyperbolic structure.