274 Curves on Surfaces, Lecture 15

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16 Orbifolds (Felikson)

Roughly speaking, a (2-dimensional) orbifold is the quotient of a surface by the action of a discrete group. These look like surfaces, possibly with boundary, some of whose points are orbifold points.

Example Consider the quotient of the disk D^2 by negation. This is just a cone; the cone point (orbifold point) is at the origin and has an angle of π.

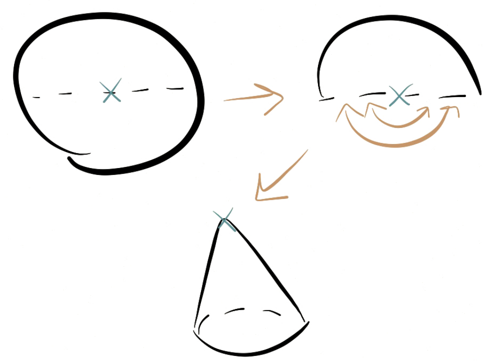

Figure 1: Quotienting a disk to obtain an orbifold.

Orbifolds are relevant to the complete statement of the classification of cluster algebras of finite mutation type. Recall that this classification was previously stated as follows:

Theorem 16.1. *(Shapiro, Tumarkin, Felikson) A cluster algebra with skew-symmetric matrix of finite mutation type is one of the following types:*

1. *Rank* 2*,*
2. *A surface cluster algebra,*
3. 11 *other exceptions.*

However, the full classification is for skew-symmetrizable matrices (matrices B such that there exists a diagonal matrix D such that BD is skew-symmetric). Now, we know that skew-symmetric matrices can be represented by quivers and quivers coming from surfaces are the block-decomposable ones.

Skew-symmetrizable matrices may be studied using certain diagrams (which do not capture everything about the matrix). We associate to an entry $b_{ij} > 0$ in such a matrix a pair of vertices i, j and an edge between them labeled $-b_{ij}b_{ji}$. These diagrams may be mutated like quivers as follows: a mutation at k

1. reverses arrows incident to k,
2. modifies triangles incident to k and their labels in such a way that $\sqrt{r} + \sqrt{r'} = \sqrt{pq}$ in the image below.

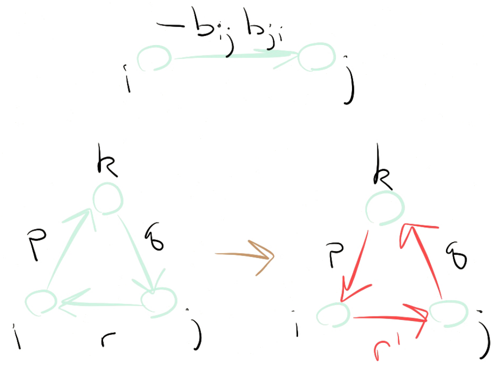

Figure 2: A diagram mutation.

The more general classification is the following.

Theorem 16.2. *(Shapiro, Tumarkin, Felikson) A cluster algebra with skew-symmetrizable matrix of finite mutation type is one of the following types:*

1. *Rank* 2*,*
2. *Obtained from blocks,*
3. $11 + 7$ *other exceptional diagrams.*

The blocks used here include the blocks we used to construct quivers coming from surfaces, but with five new blocks. They may be thought of as coming from triangulations of orbifolds.

We require all of the orbifold points to be cone points of angle π. A triangulation of such an orbifold is a maximal compatible set of arcs, where an arc may have endpoints either marked points or orbifold points, and compatibility means non-crossing and

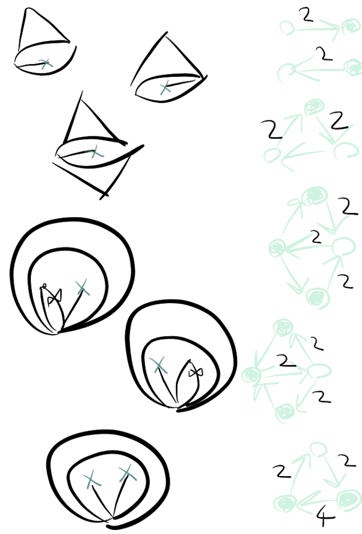

Figure 3: Blocks and the triangulations of orbifolds from which they arise.

also means that we forbid two arcs from having endpoint the same orbifold point. We allow our arcs to be tagged on ends which are not attached to orbifold points (since orbifold points do not have horocycles associated to them).

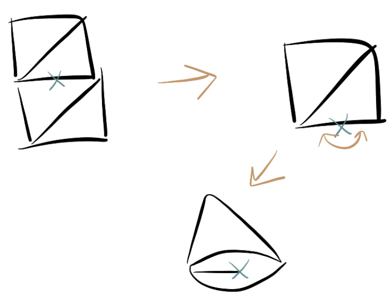

Figure 4: Quotienting a triangulation to obtain an orbifold triangulation.

We also need to decide what flips between triangulations are for arcs with an orbifold endpoint. These are obtained by removing the arc and drawing the only other compatible arc.

Theorem 16.3. *Flips act transitively on tagged triangulations of orbifolds.*

We can build diagrams from tagged triangulations of orbifolds in the same way as before, except that edges coming to or from an arc with an orbifold endpoint should be labeled by a 2 (and edges coming both to and from an arc with an orbifold endpoint should be labeled by a 4). Mutation of diagrams then corresponds to flips of triangulations, and any diagram we obtain in this way decomposes into blocks.

We cannot reconstruct a skew-symmetrizable matrix from a diagram because we have lost information in only labeling by $-b_{ij}b_{ji}$. This can be fixed by assigning weights to vertices given by the corresponding entries of the diagonal matrix D symmetrizing the matrix.

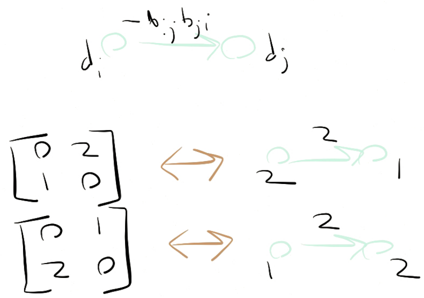

Figure 5: Using weights on vertices to reconstruct matrix entries.

In a block decomposition with weights, all white vertices necessarily have the same weight w, and all black vertices have weight either $2w$ or $\frac{w}{2}$.

Hence we should use weighted orbifolds, namely orbifolds such that their orbifold points are marked either 2 or $\frac{1}{2}$. These are the geometric objects associated to the second type of cluster algebra in the theorem above. From such data we can choose a triangulation, then build a weighted diagram, from which we can recover a skew-symmetrizable matrix.

We would now like to introduce hyperbolic structure (so we can assign λ-lengths, etc.). We first restrict our attention to the case that all of the orbifold points have weight $\frac{1}{2}$. We make all ordinary triangles ideal hyperbolic triangles, and we obtain orbifold points by symmetrically gluing ideal hyperbolic triangles (dropping an angle bisector). We may now decorate with horocycles (keeping in mind that there are no horocycles around orbifold points), and we define λ-lengths as before, with the λ-length of an arc with an orbifold endpoint the λ-length of its lift to a symmetrically glued ideal hyperbolic triangle.

The Ptolemy relation can look different here.

To accommodate orbifold points of weight 2, it is necessary to replace them by special marked points. These marked points are special because they assigned a self-conjugate horocycle.

With the above convention, λ-lengths parameterize the Teichmüller space associated to the orbifold.

Example The quiver B_n can be realized using one orbifold point. It is a quotient of D_n.

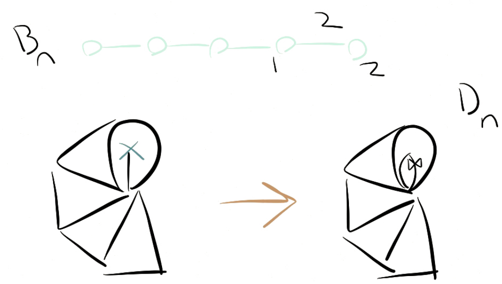

Figure 6: The B_n quiver as a quotient of the D_n quiver.

Example The quiver C_n can be realized using one orbifold point. It is a quotient of A_{2n-1}.

It is a natural question to ask whether any orbifold can be realized as a quotient of an ordinary surface. If there are an even number of orbifold points of weight $\frac{1}{2}$, we can pair them up and cut along lines connecting them. If there is more than one orbifold point of weight $\frac{1}{2}$, then we can pair up two of them and cut, and this doubles the remaining number of orbifold points. If the boundary is nonzero, we can cut along a line connecting an orbifold point and a point on the boundary. Finally, if our surface is not S^2, we can cut along a non-contractible loop.

The remaining case is when the surface is S^2 with one orbifold point of weight $\frac{1}{2}$; this orbifold cannot be realized as a quotient.

Laminations and skein relations work well in this formalism.

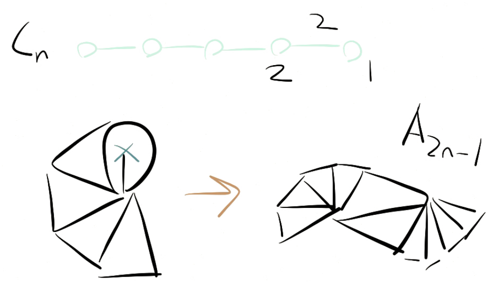

Figure 7: The C_n quiver as a quotient of the A_{2n-1} quiver.