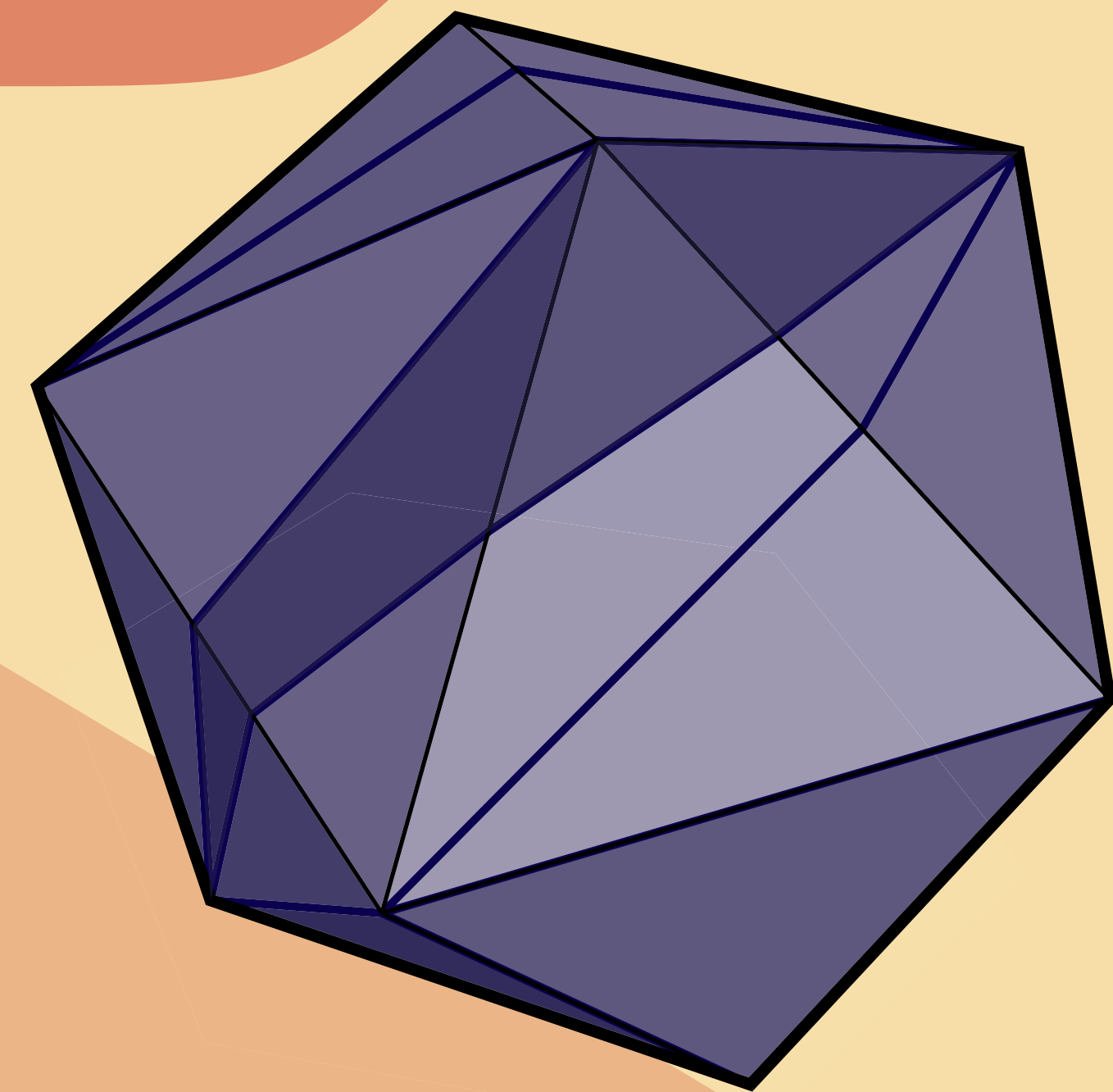
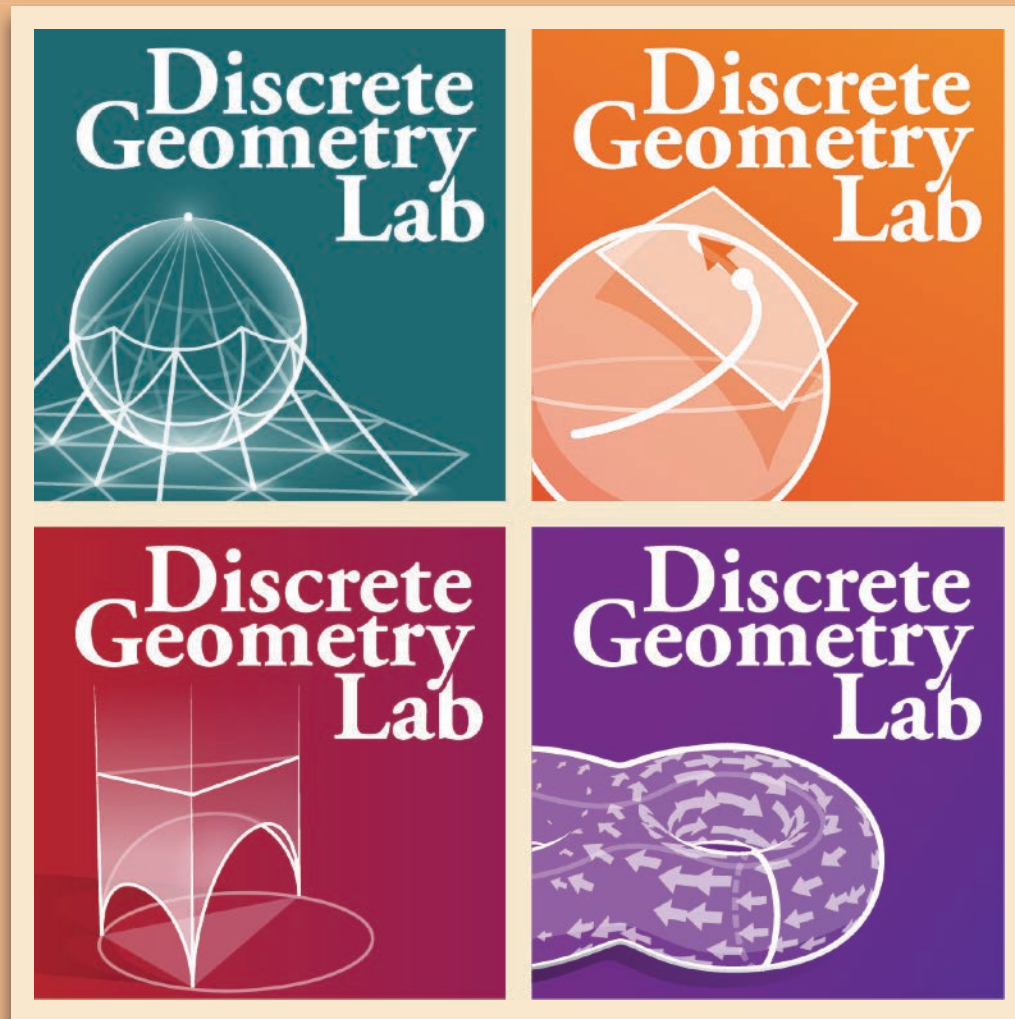


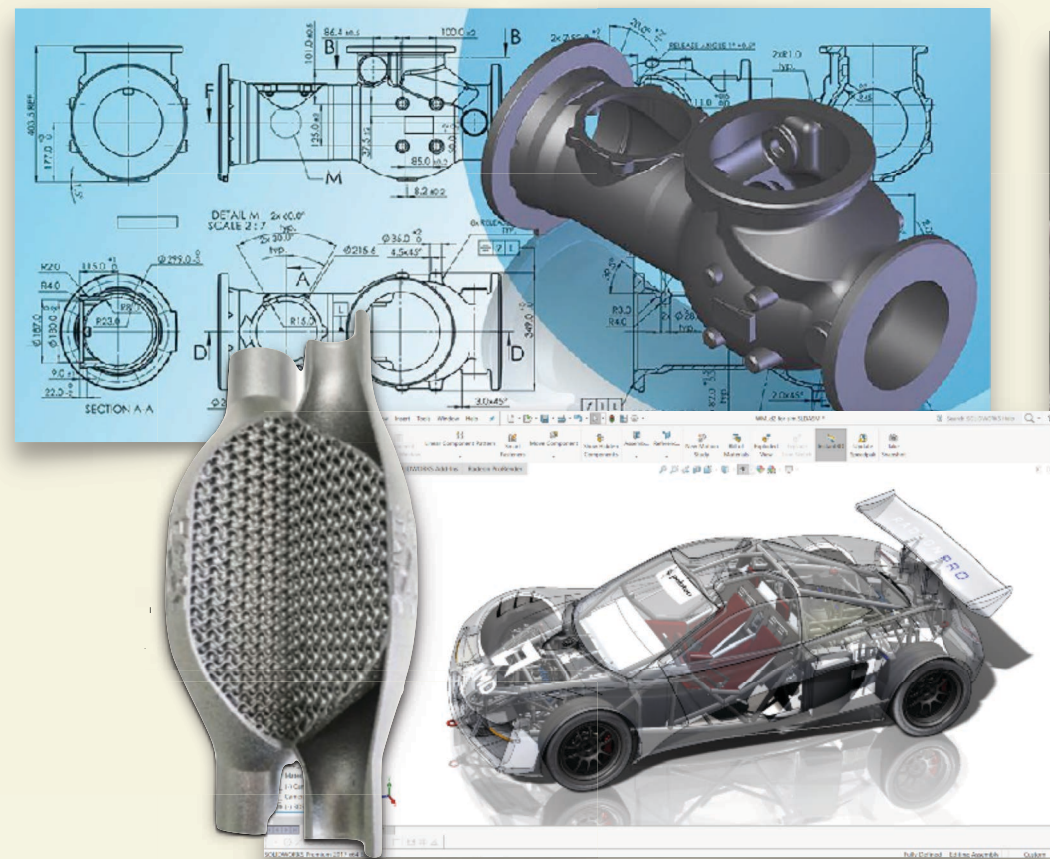
Intrinsic Triangulations in Geometry Processing

Mark Gillespie, ASSISTANT PROFESSOR

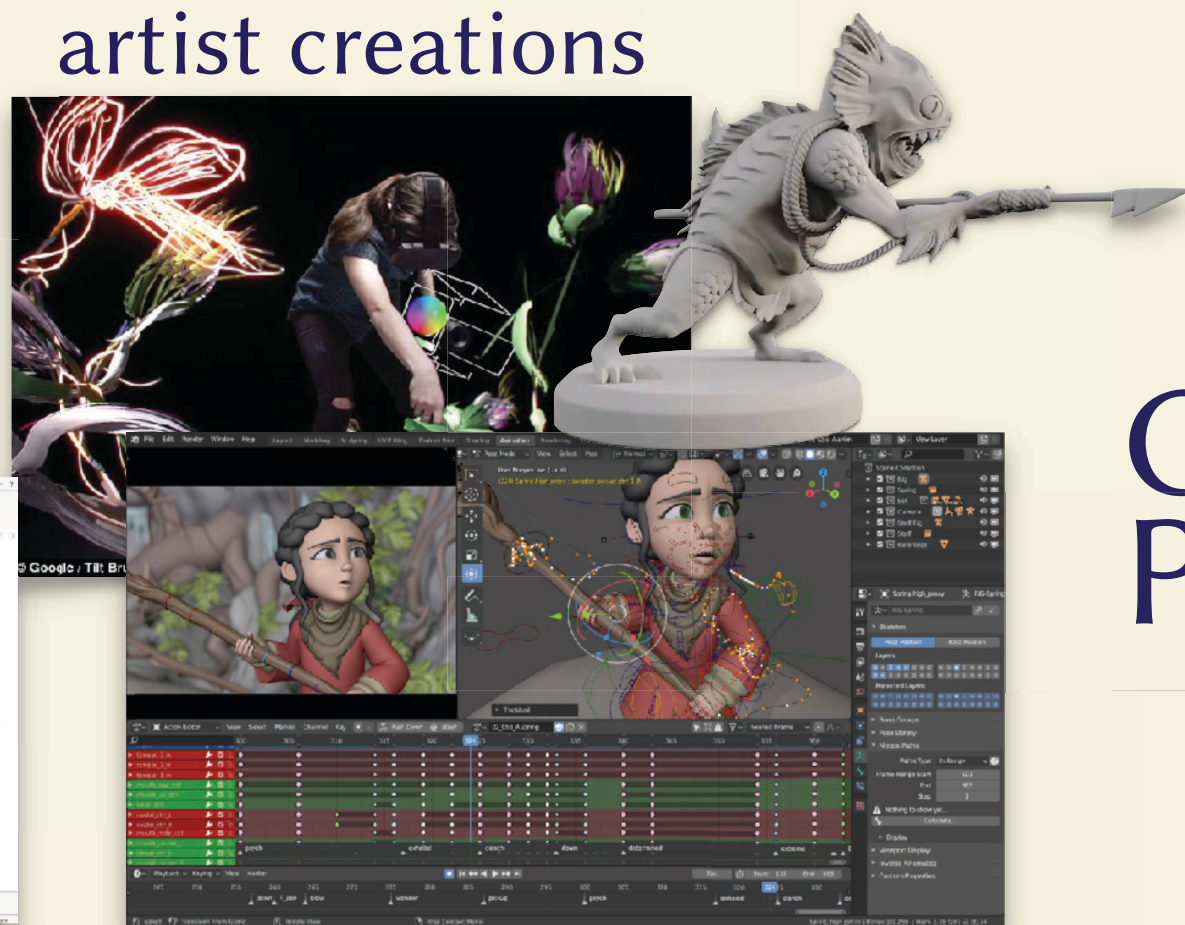


Geometry is all around us

technical assets



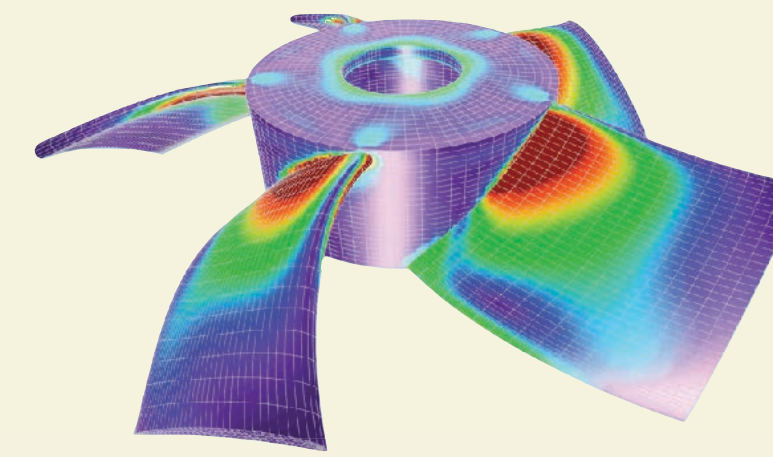
artist creations



Geometry Processing

design algorithms to be:

- simple
- efficient
- correct



simulation



robotics



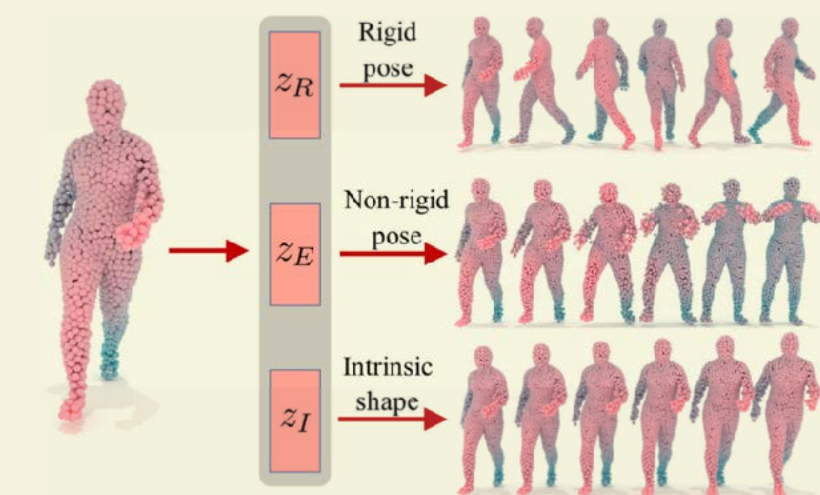
creative tools



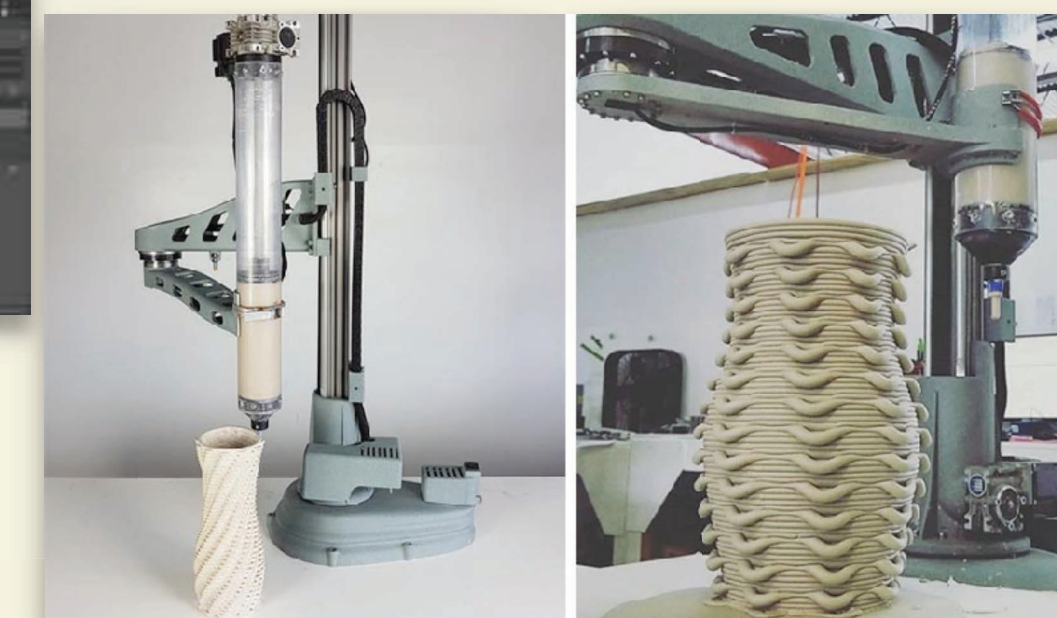
animation



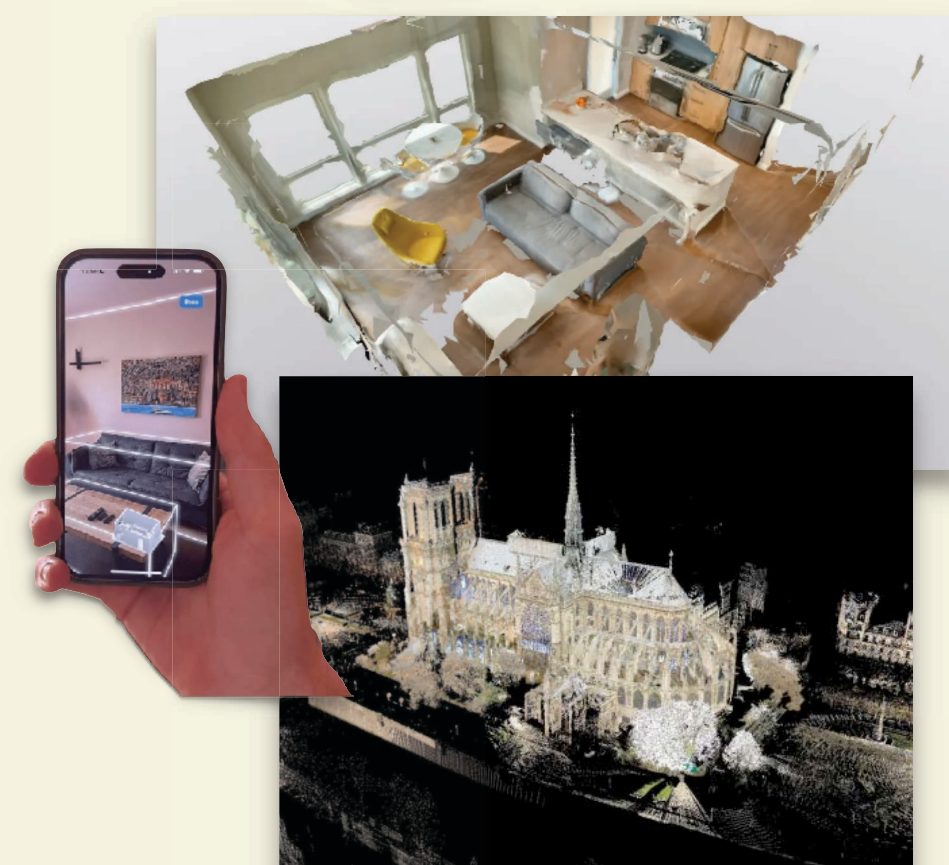
medical imaging



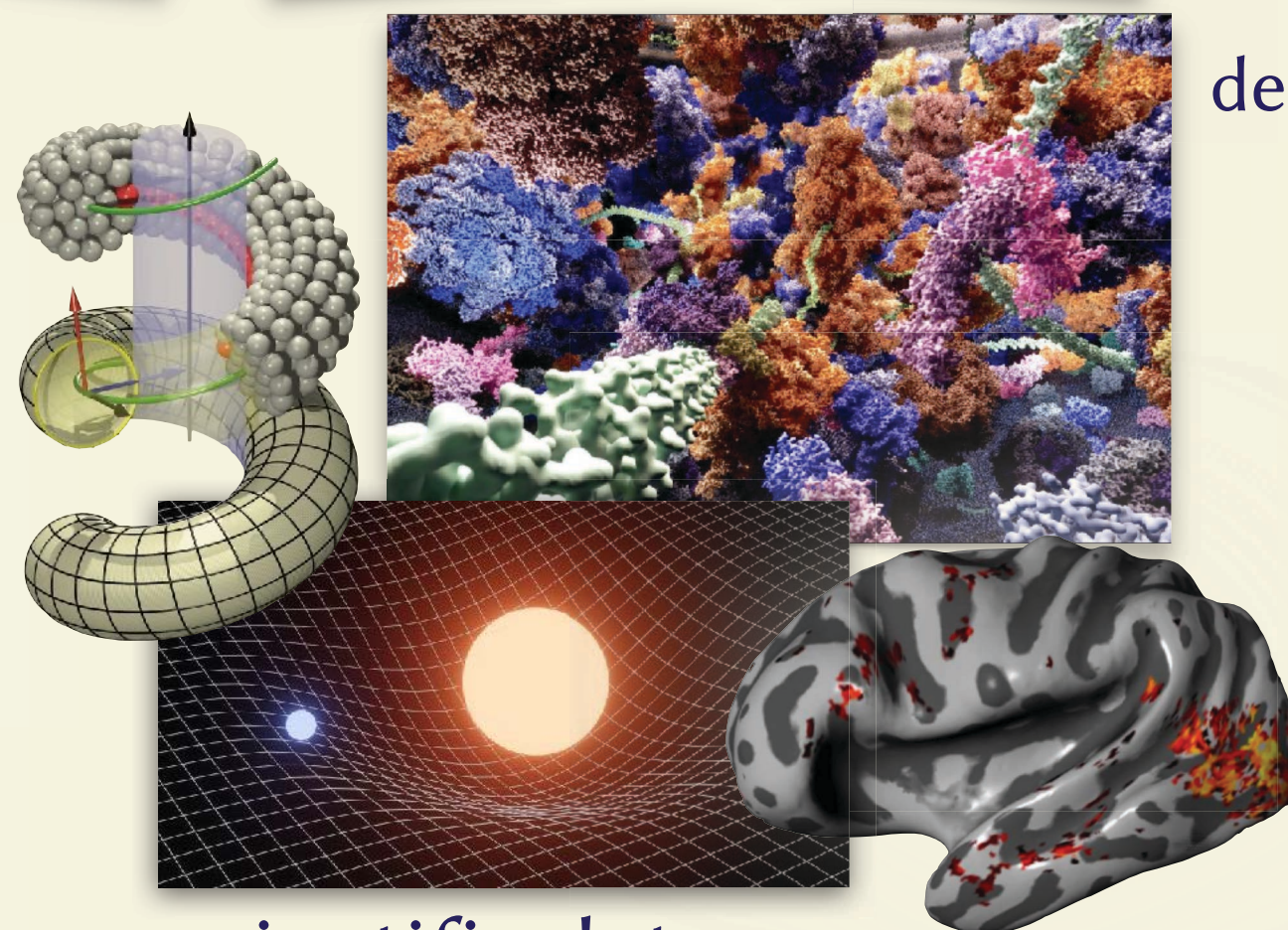
geometric ML



digital fabrication



3D scans



scientific data

Triangle meshes

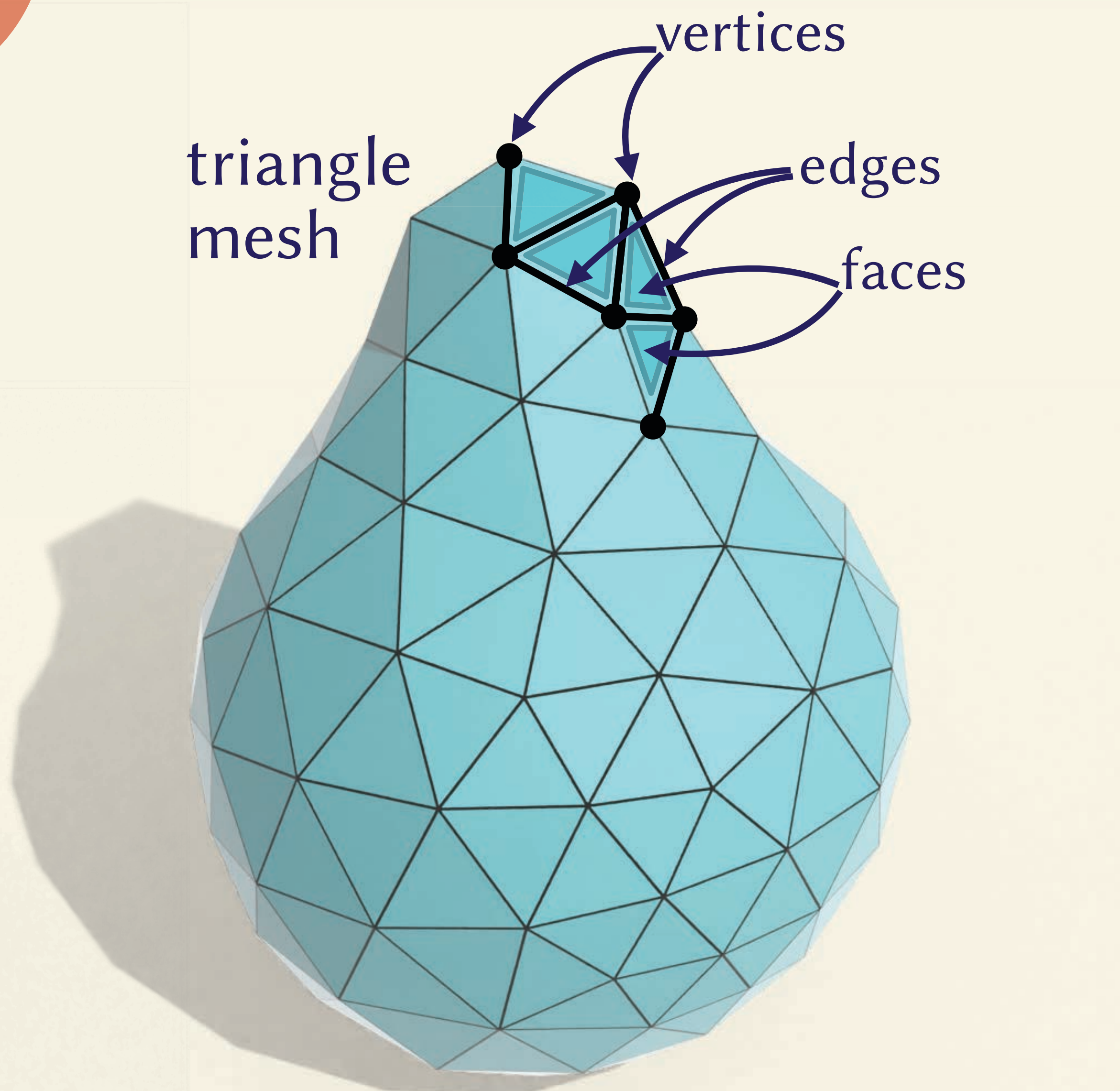
smooth
surface

triangle
mesh

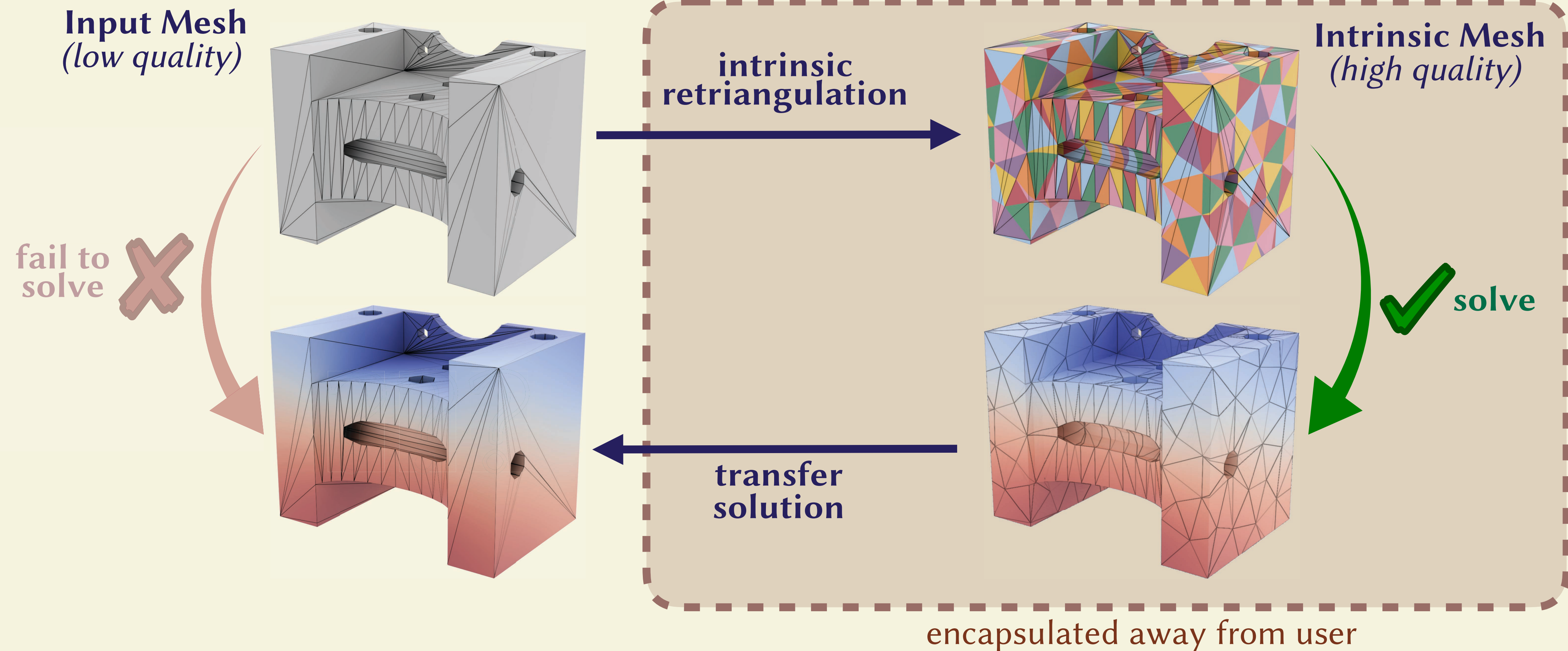
vertices

edges

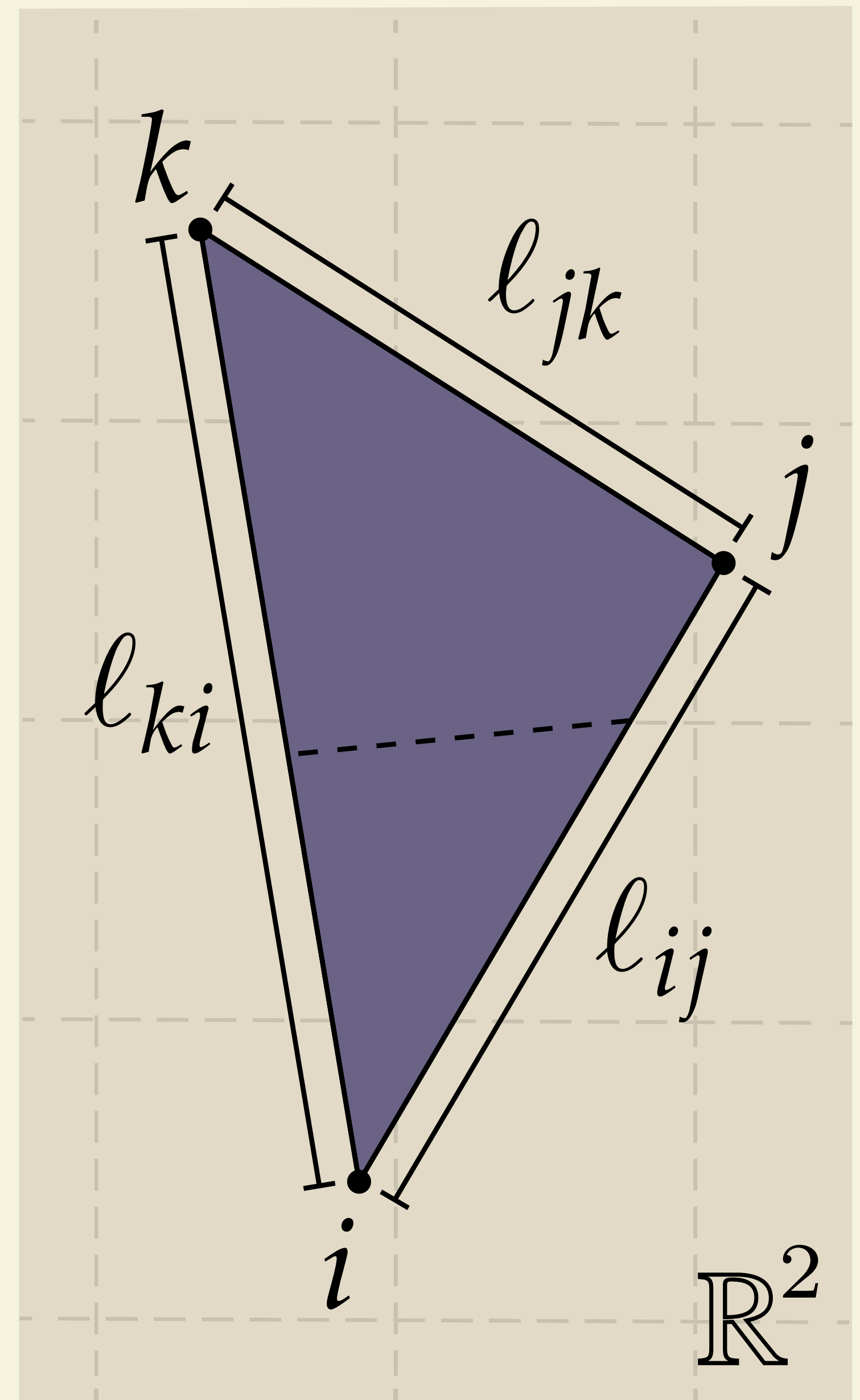
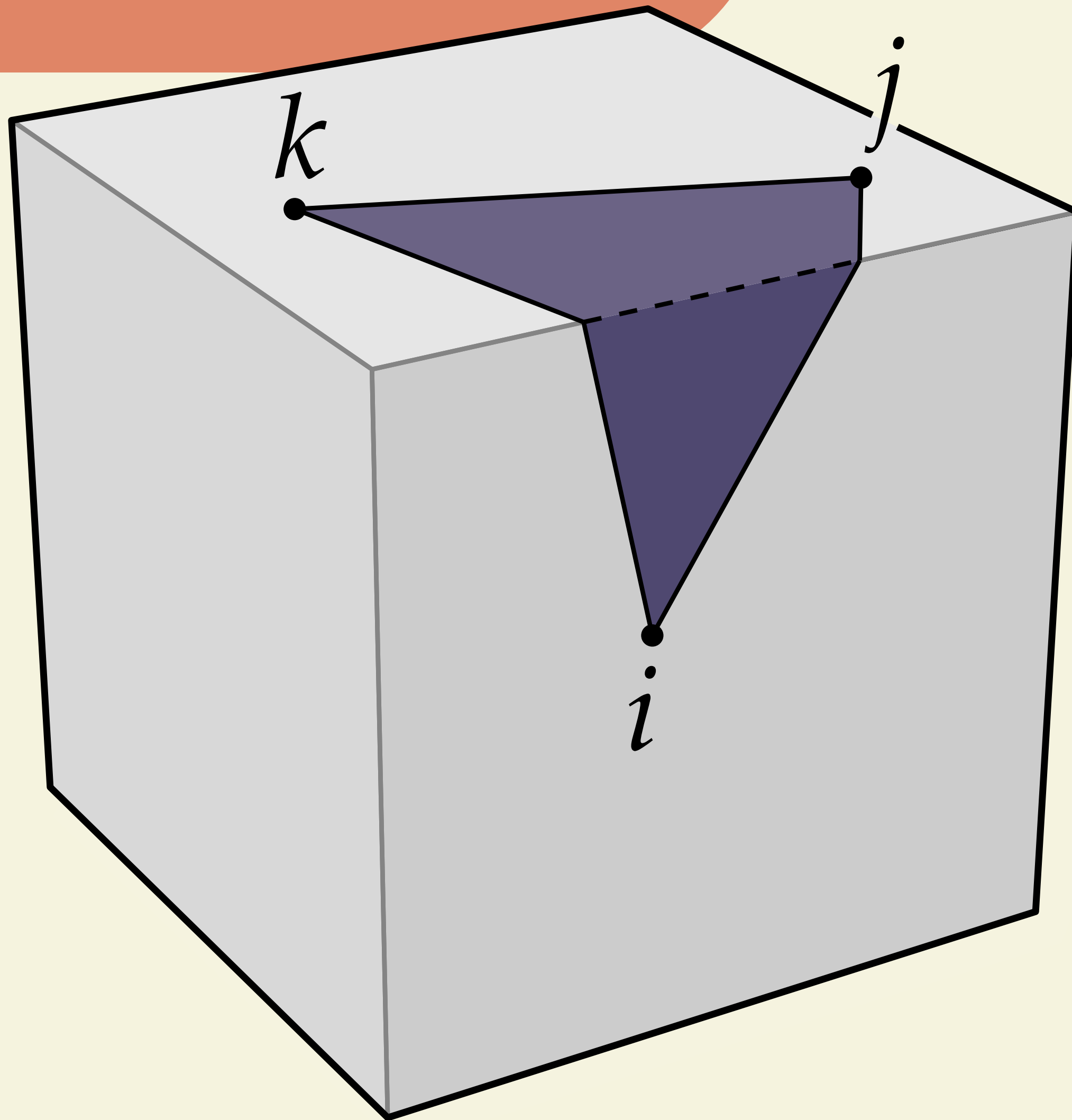
faces



Working with low-quality triangle meshes

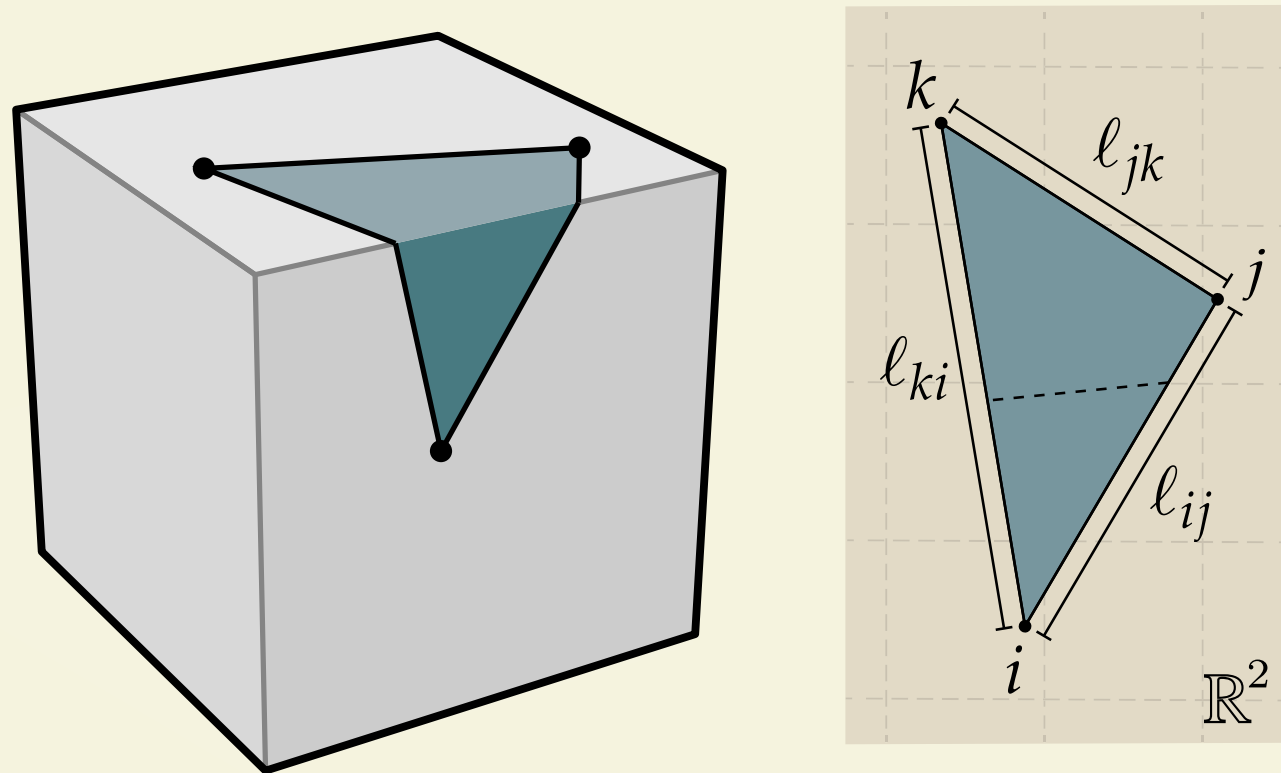


Intrinsic triangles

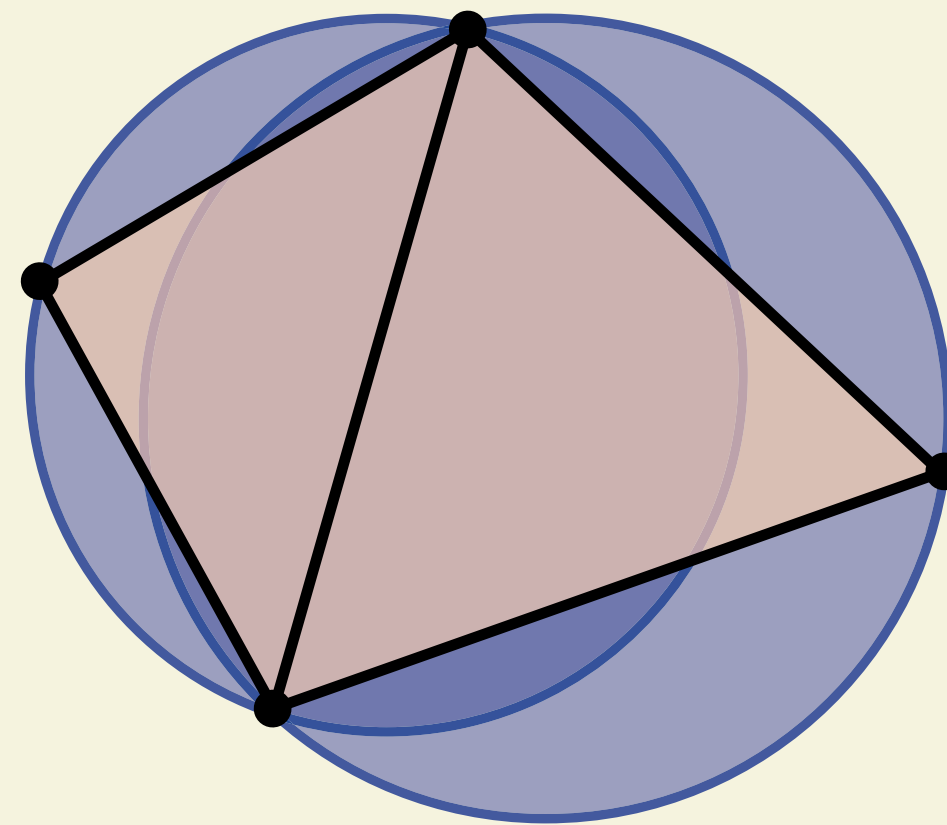


Outline

I. Preliminaries

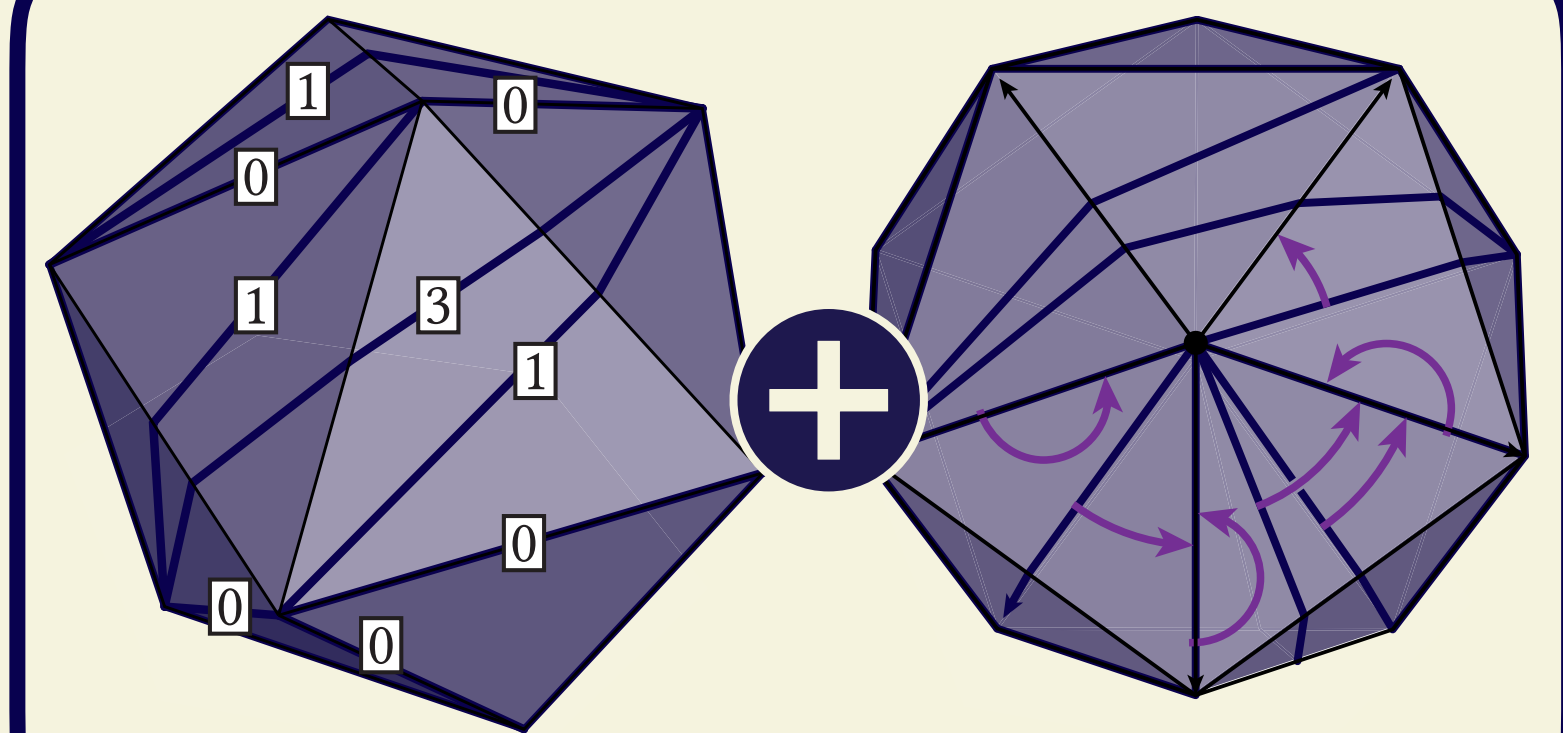


II. Delaunay triangulations



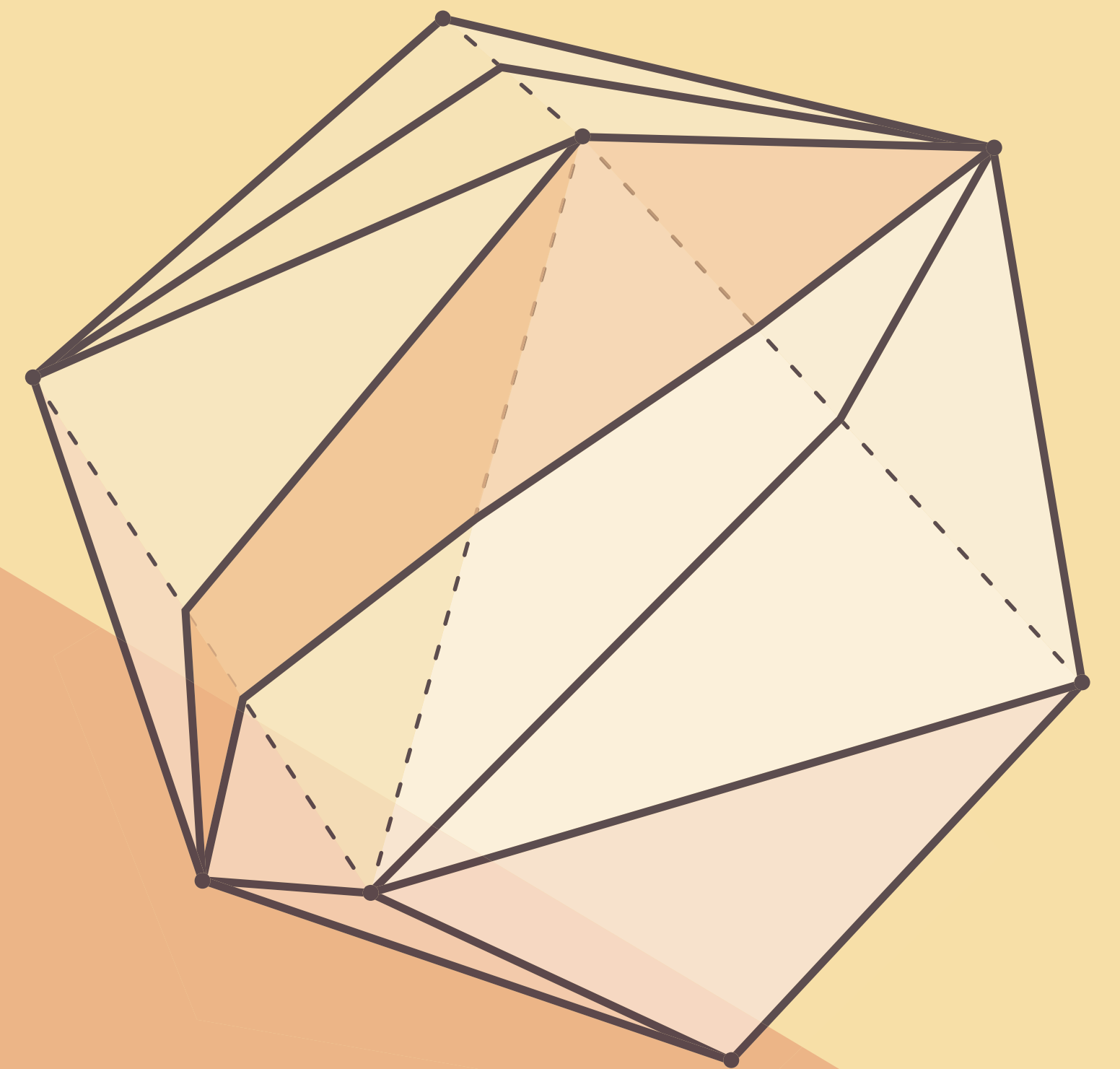
[Sharp, G. & Crane. 2021. Geometry Processing with Intrinsic Triangulations. *SIGGRAPH '21 Courses*]

III. Data structures for intrinsic triangulations

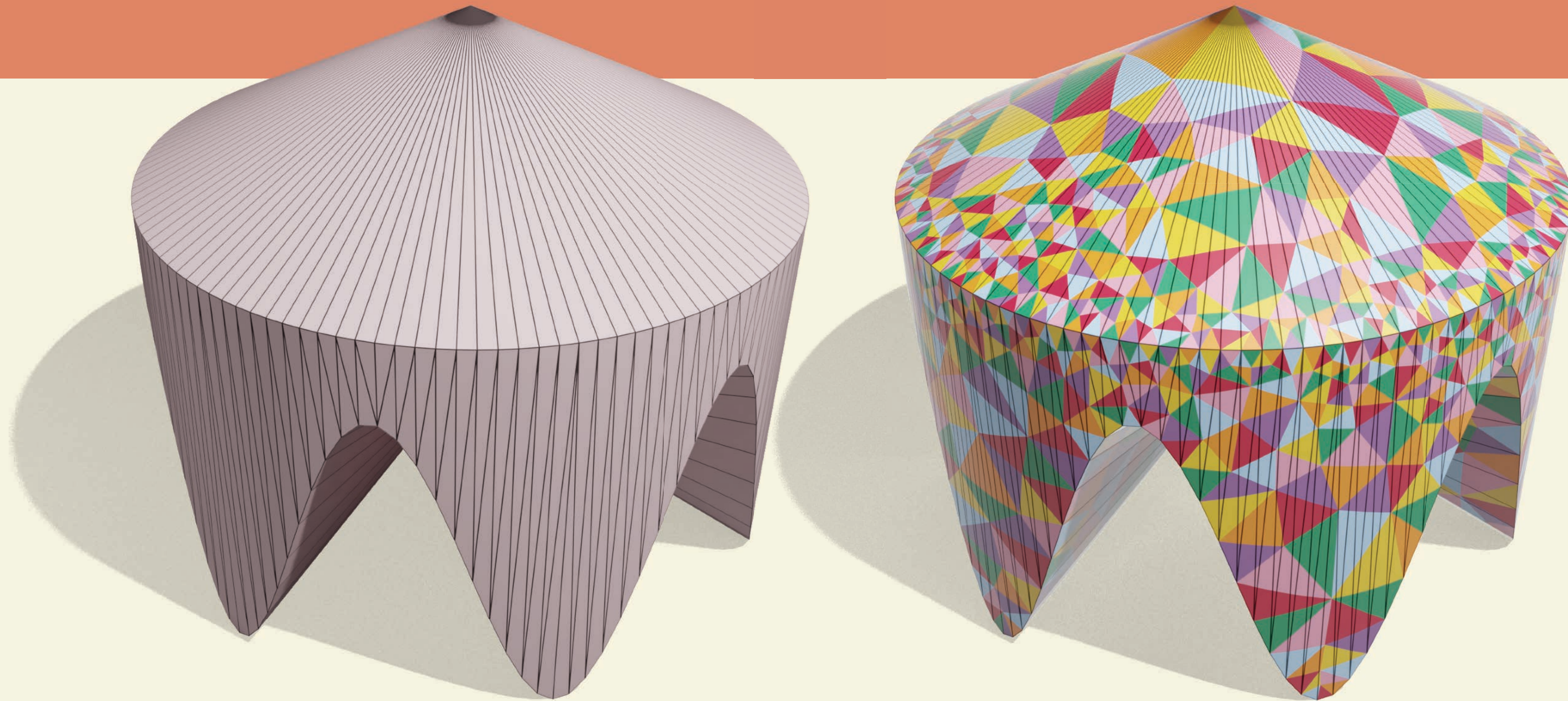


[G., Sharp, & Crane. 2021. Integer Coordinates for Intrinsic Geometry Processing. *ACM Trans. Graph.*]

I. Preliminaries



A brief history of intrinsic triangulations

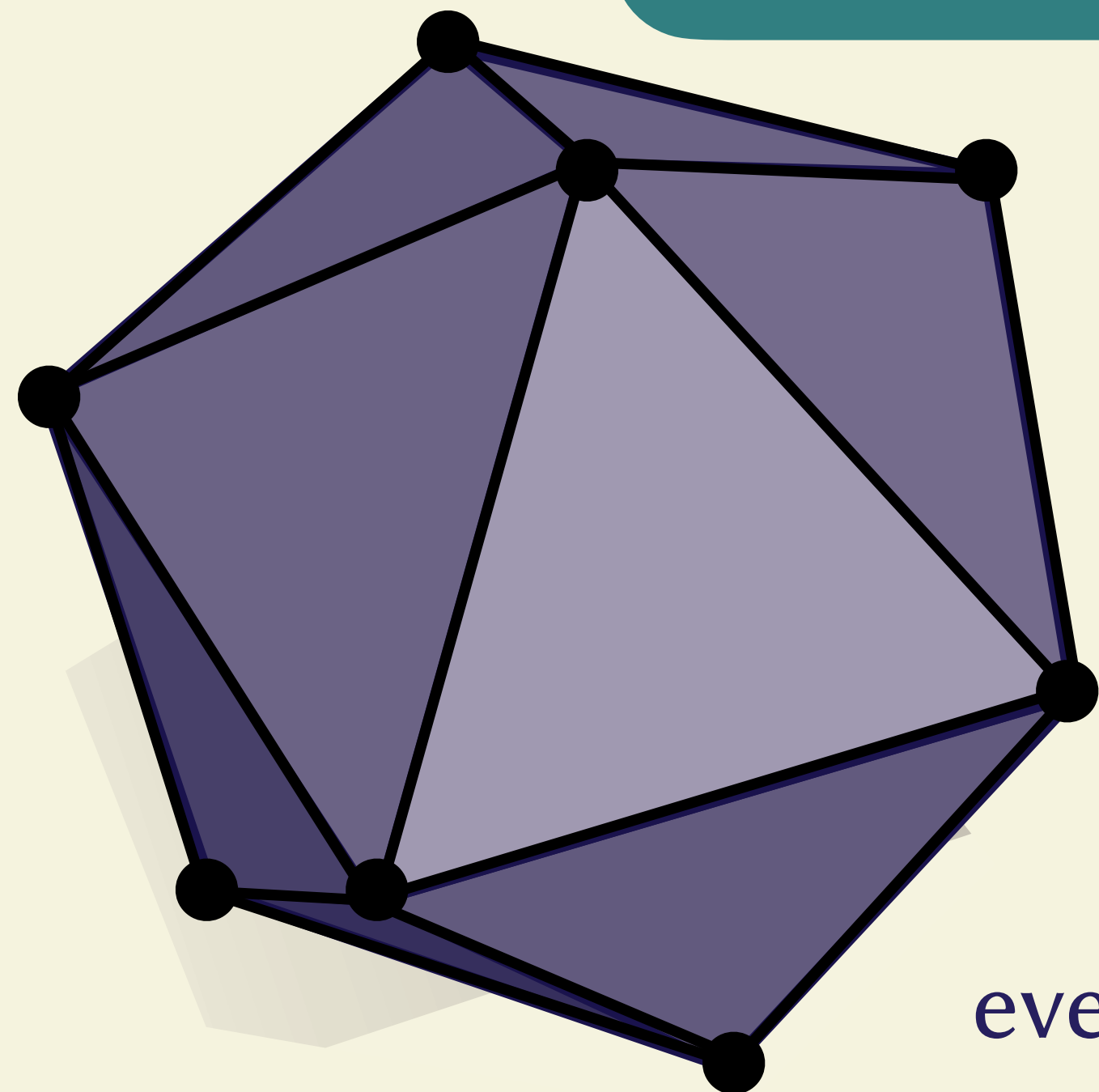


Foundations: [Alexandrov 1948; Regge 1961]

Geometry Processing: [Fisher, Springborn, Bobenko & Schröder 2006; Bobenko & Springborn 2007, Bobenko & Izmestiev 2008; Sun, Wu, Gu & Luo 2015; Sharp, Soliman & Crane 2019; Fumero, Möller & Rodolà 2020; G., Springborn & Crane 2021; Finnendahl, Schwartz & Alexa 2023; Liu *et al.* 2023]

Triangulations

cell complex



vertex set V
edge set E
face set F

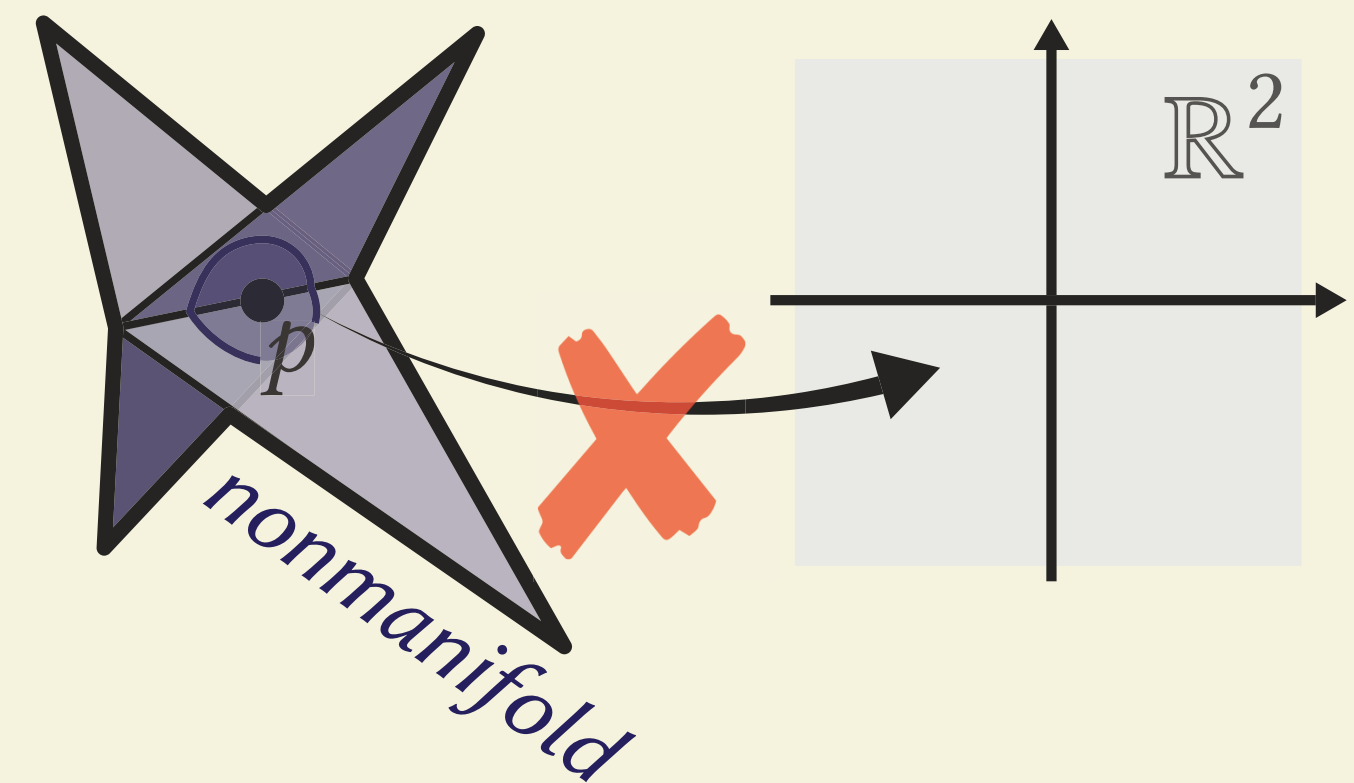
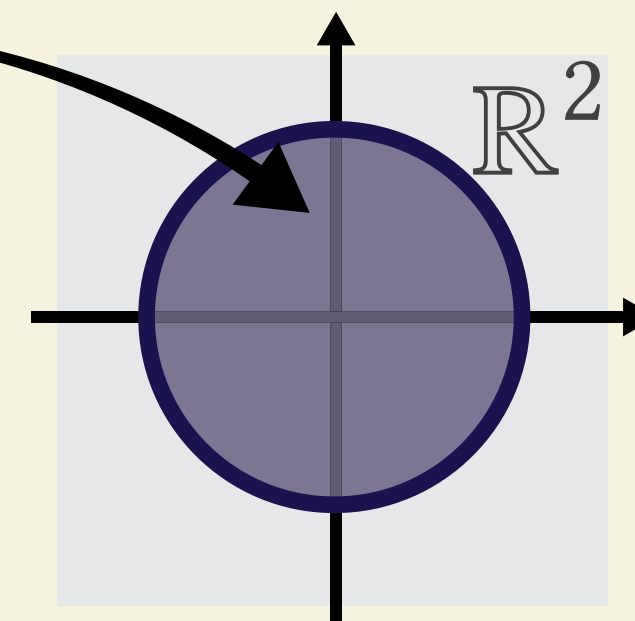
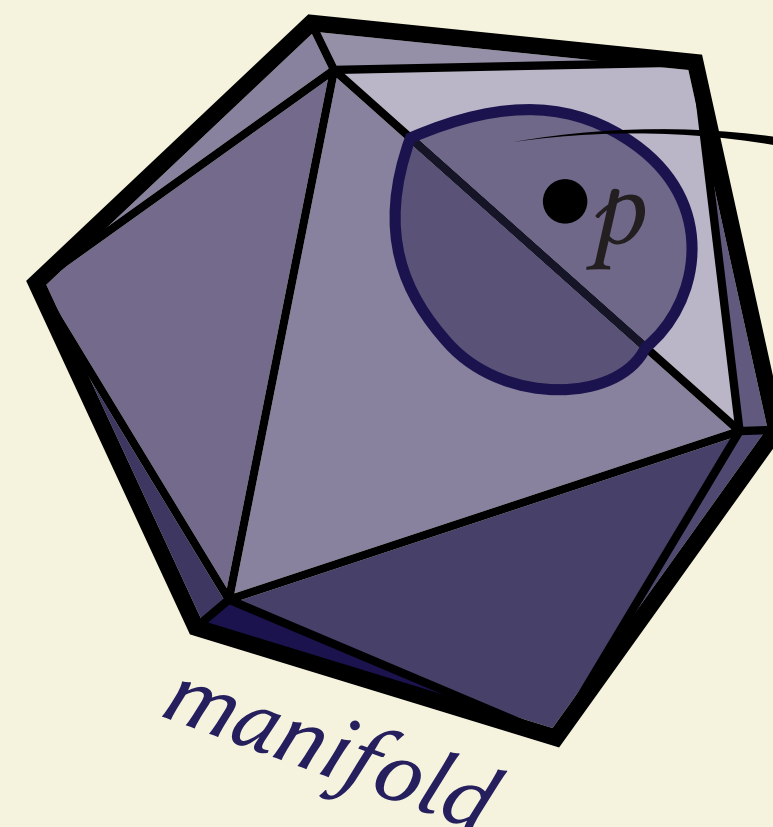
every face has
3 edges

Definition

A *surface triangulation* is a manifold
2-dimensional cell complex whose
faces are all triangles

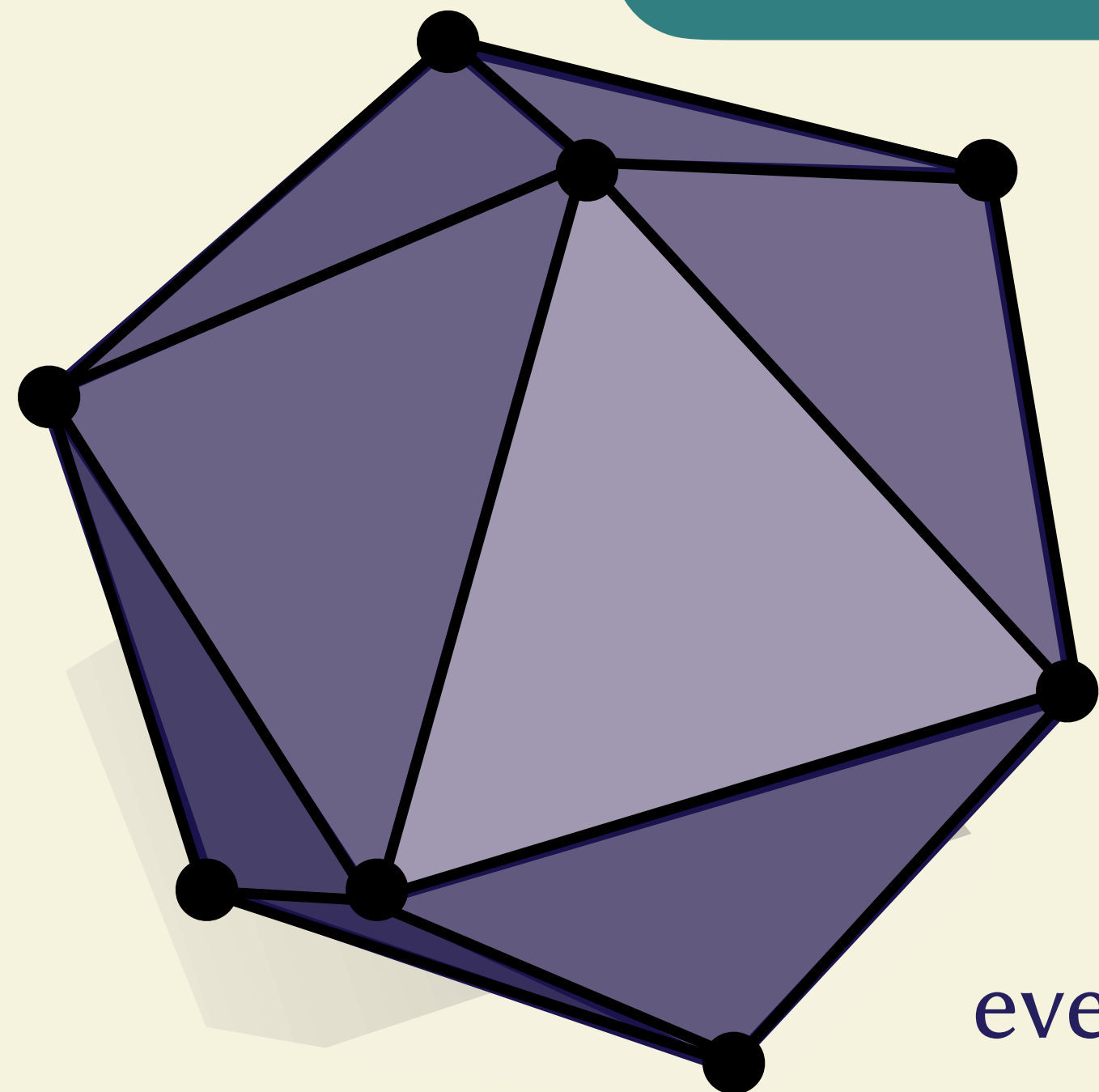
manifold

each point has a neighborhood
that looks like the plane



Triangulations

cell complex



vertex set V
edge set E
face set F

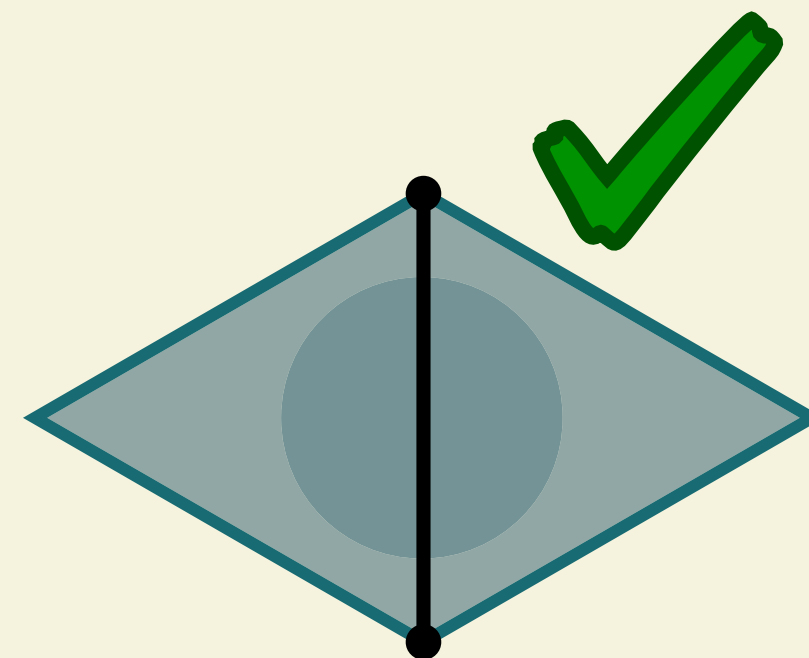
every face has
3 edges

Definition

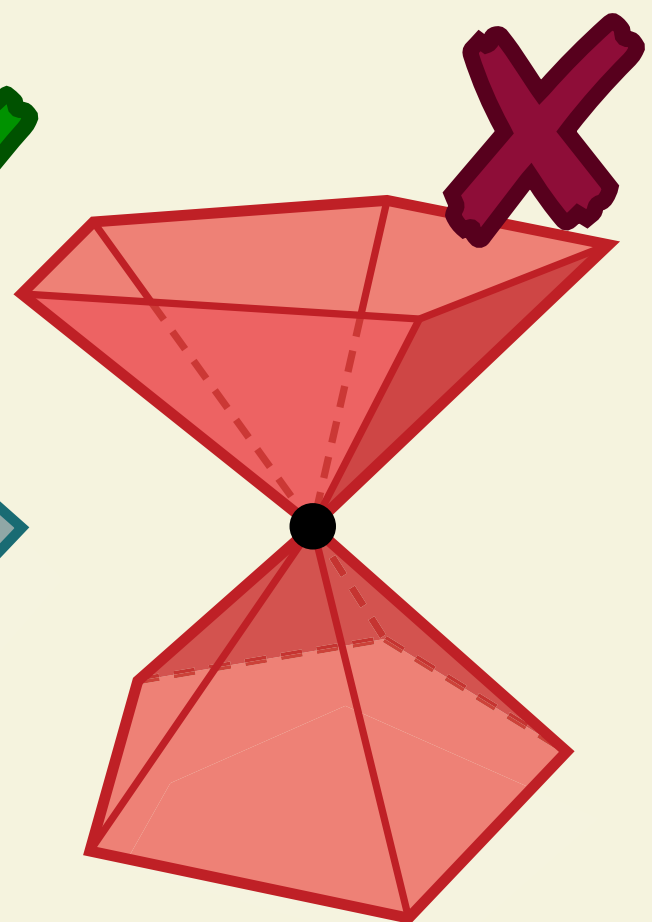
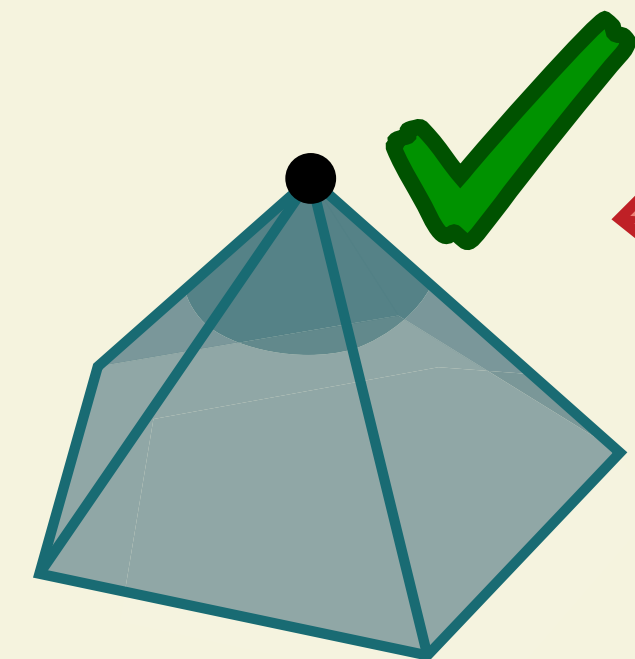
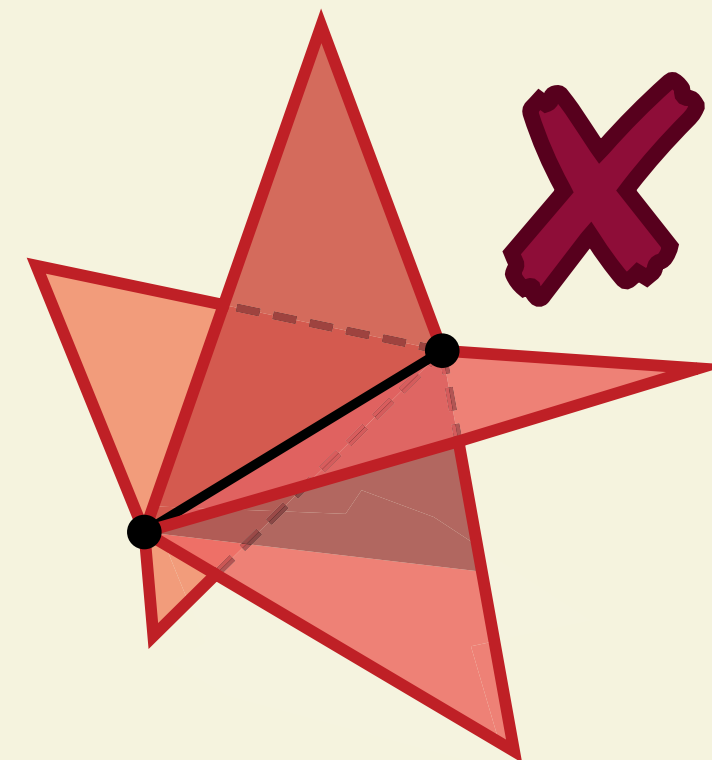
A *surface triangulation* is a **manifold** 2-dimensional cell complex whose faces are all triangles

manifold

each point has a neighborhood that looks like the plane



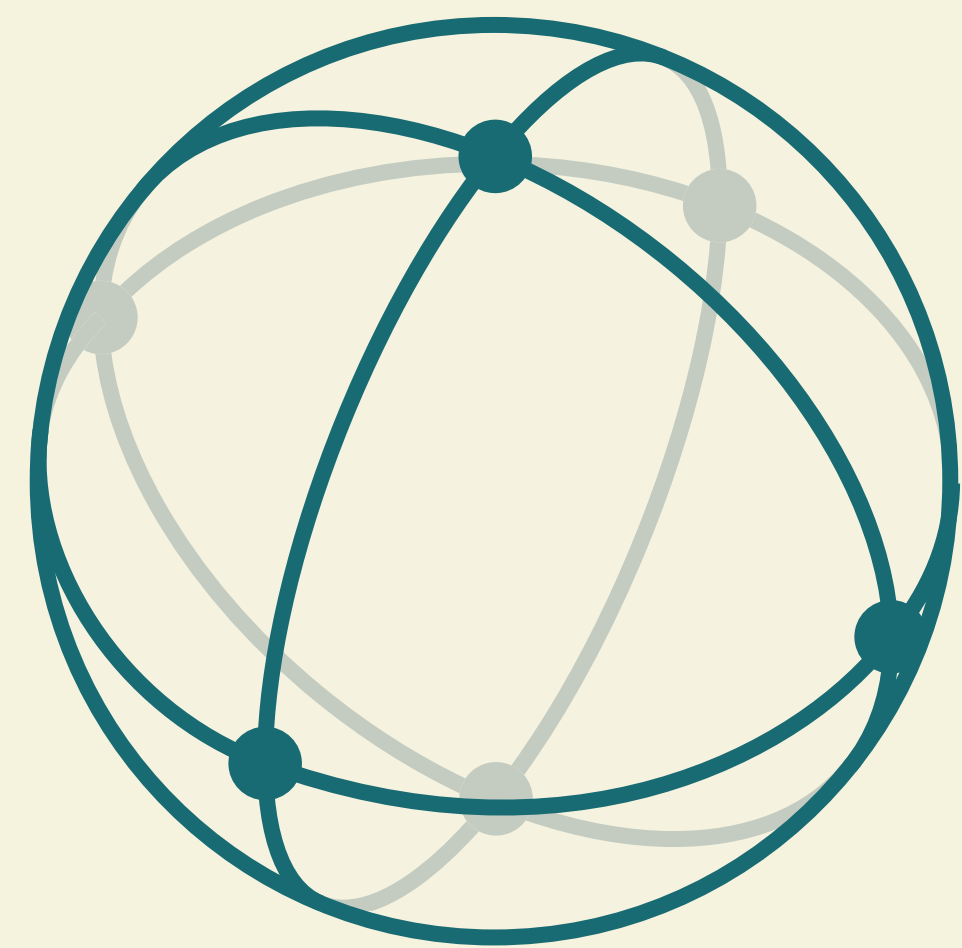
each edge is in 1 or 2 faces
(but no more)



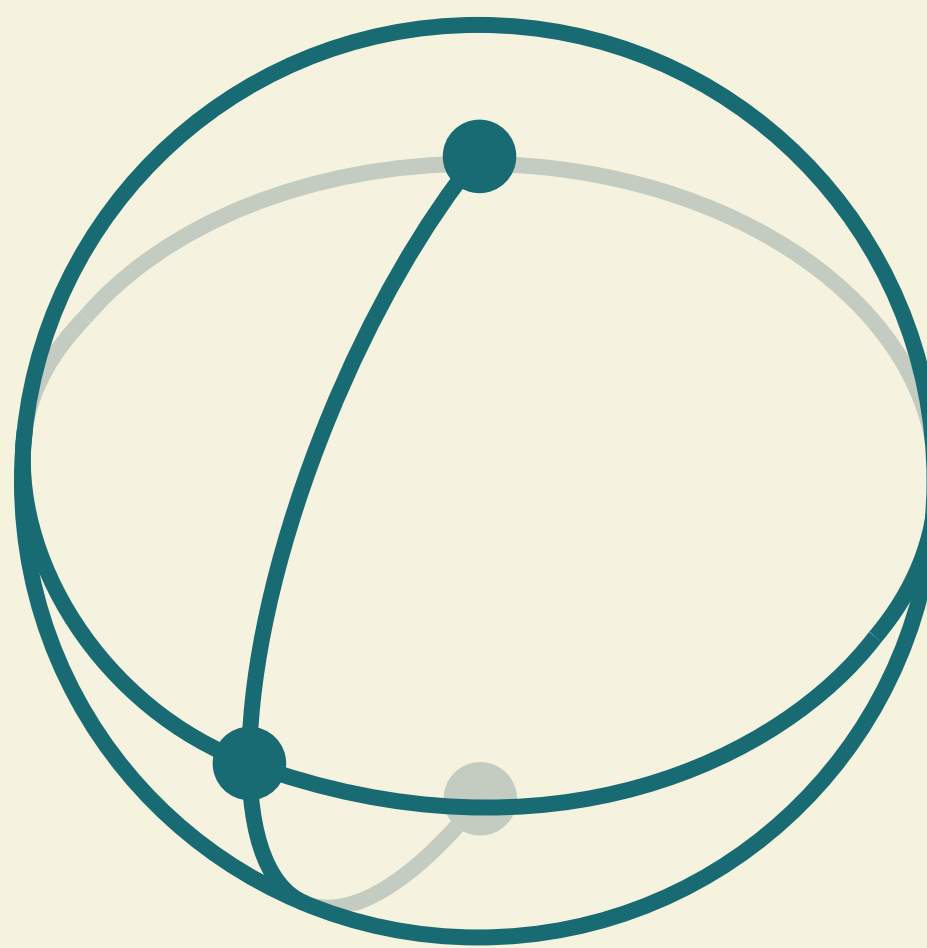
the faces around a vertex
form a single cycle

Triangulations

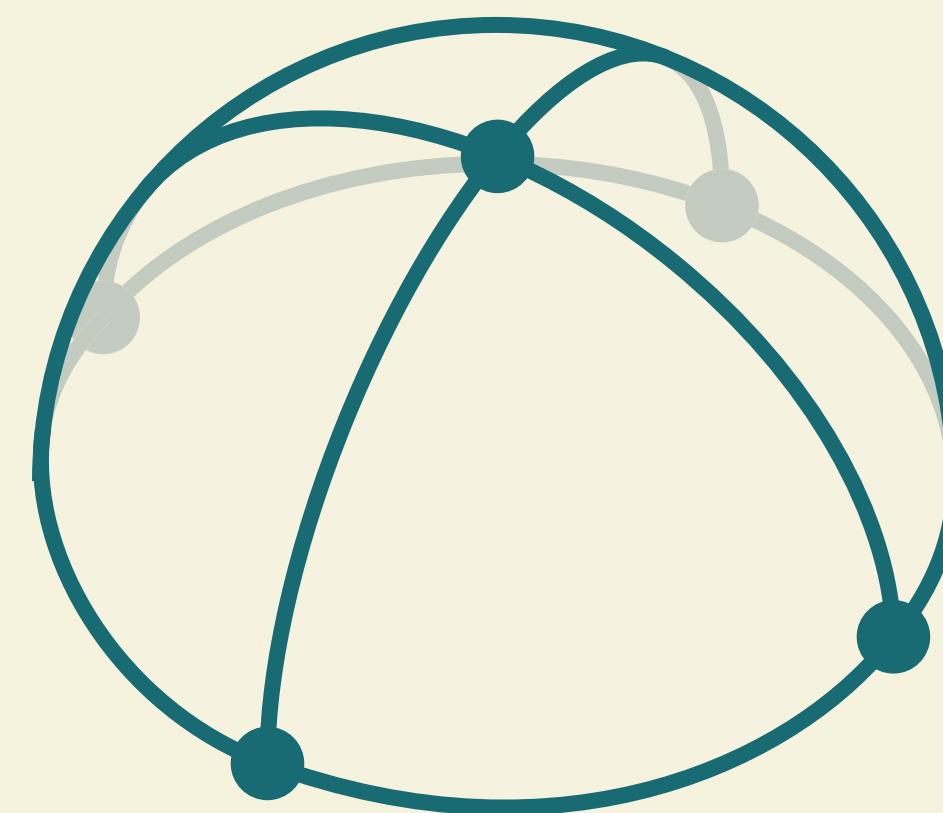
still allow for lots of interesting behavior



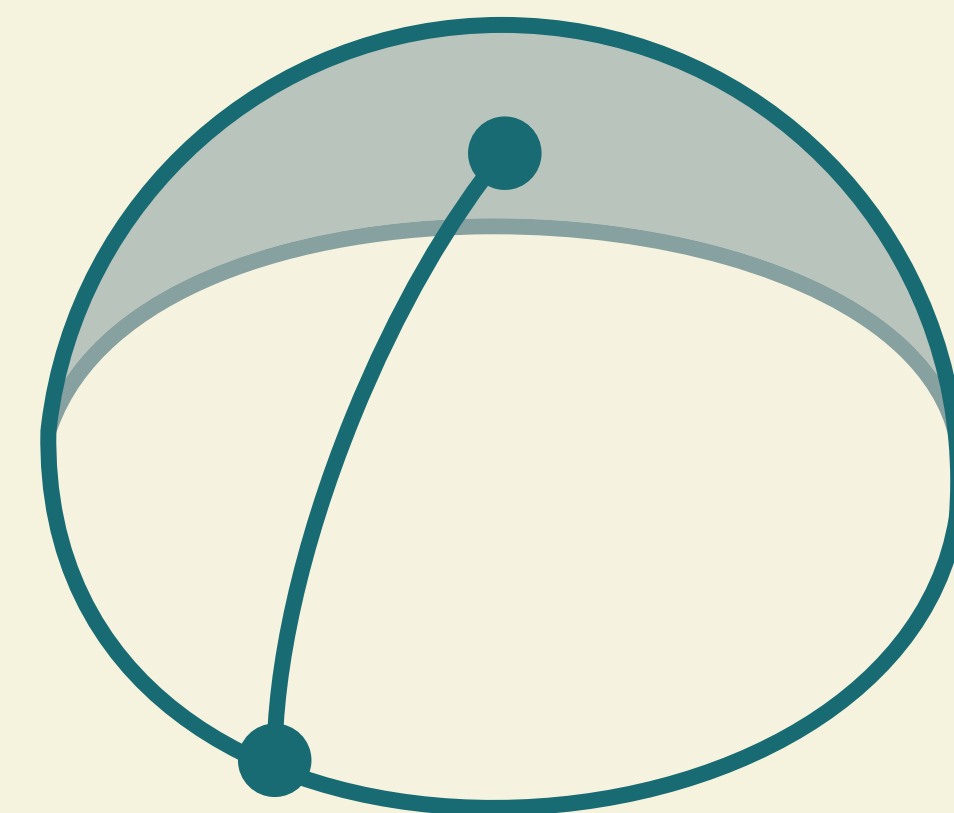
may have
multiple edges
between two
vertices



may have
edges from a
vertex to itself



may have
boundaries

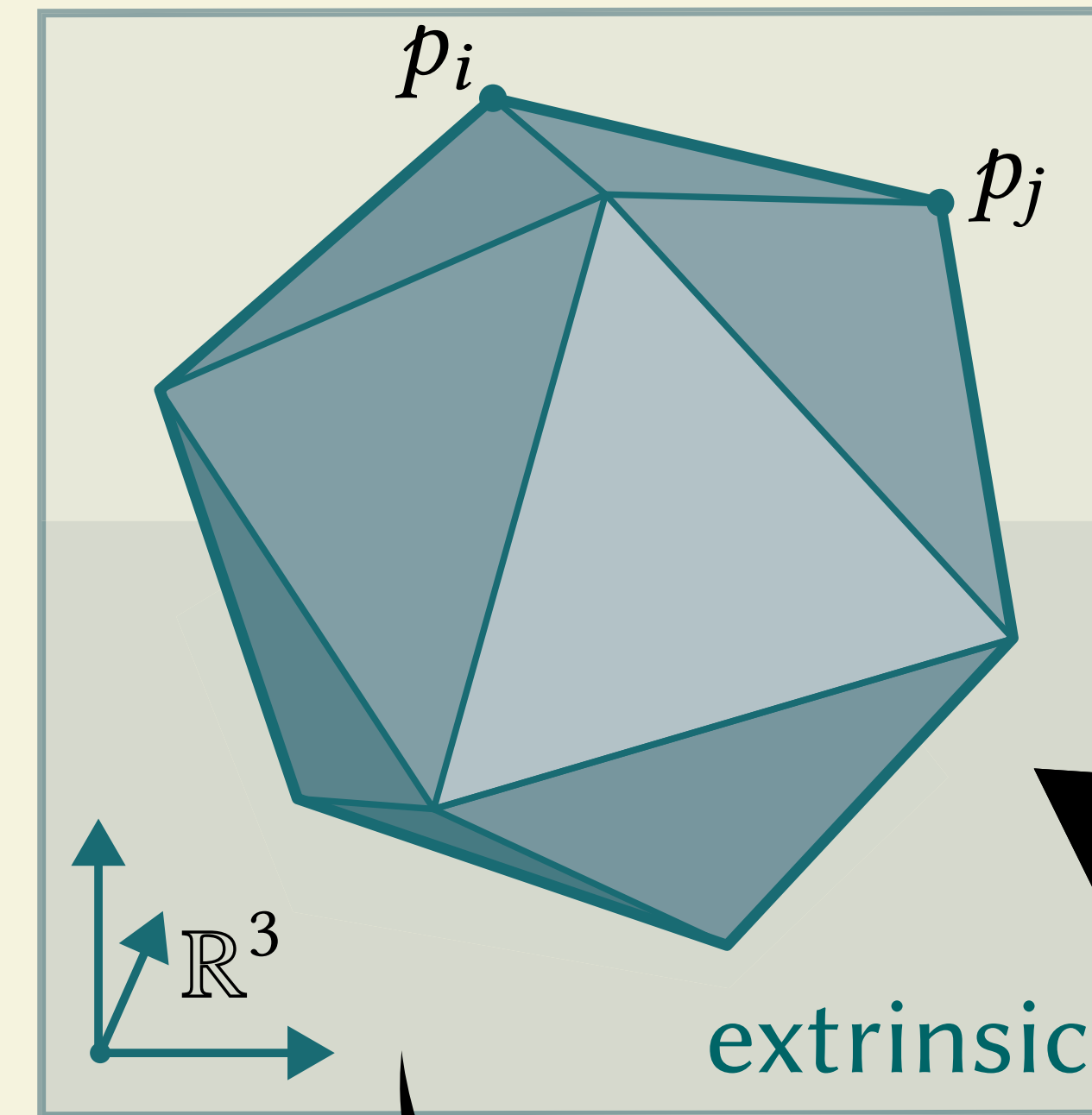


and any
combination

Extrinsic and intrinsic triangulations

Definition

An *extrinsic triangulation* is a triangulation equipped with vertex positions $p : V \rightarrow \mathbb{R}^3$



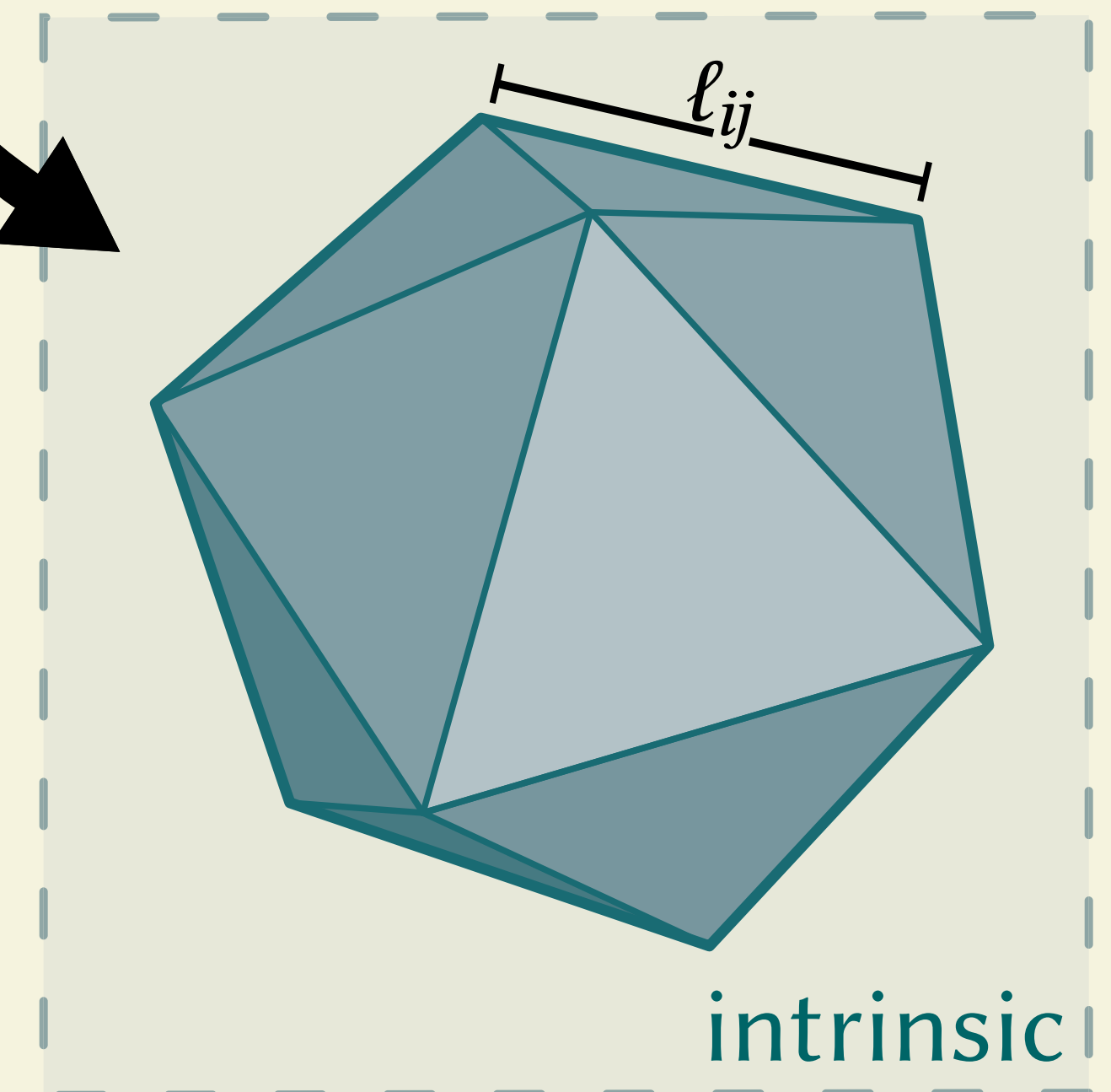
(in convex case, see
[Bobenko &
Izmestiev 2008])

hard

easy

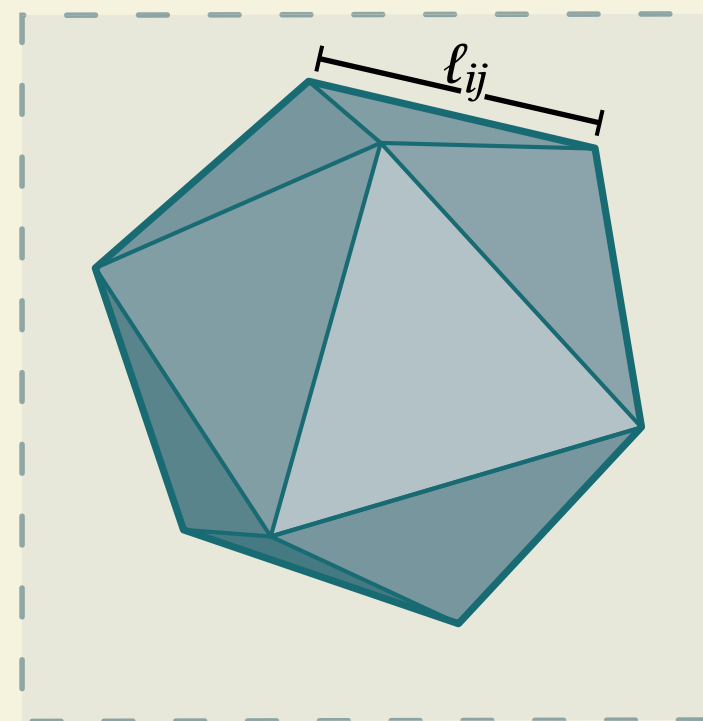
Definition

An *intrinsic triangulation* is a triangulation equipped with positive edge lengths $\ell : E \rightarrow \mathbb{R}_{>0}$ satisfying the triangle inequality

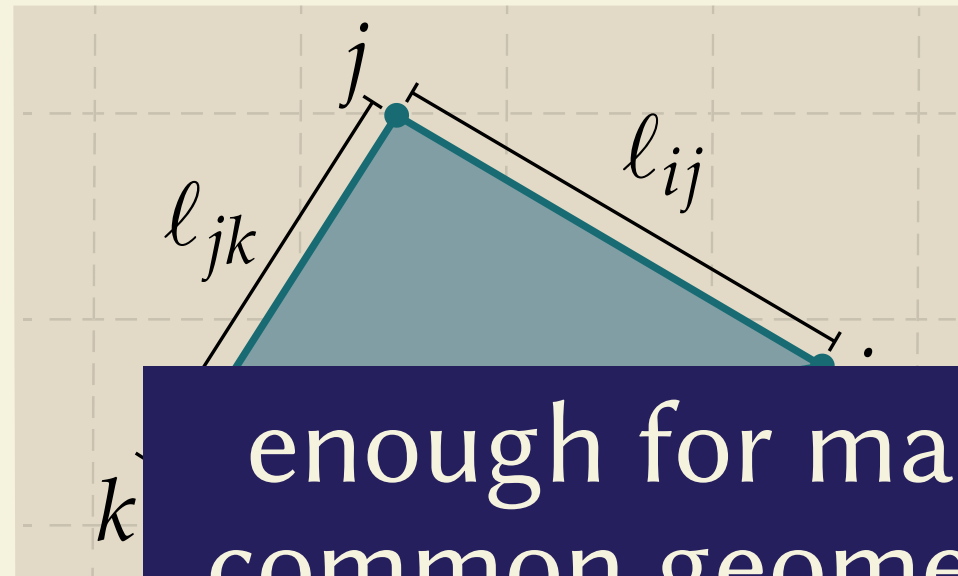


Extrinsic and intrinsic triangulations

intrinsic



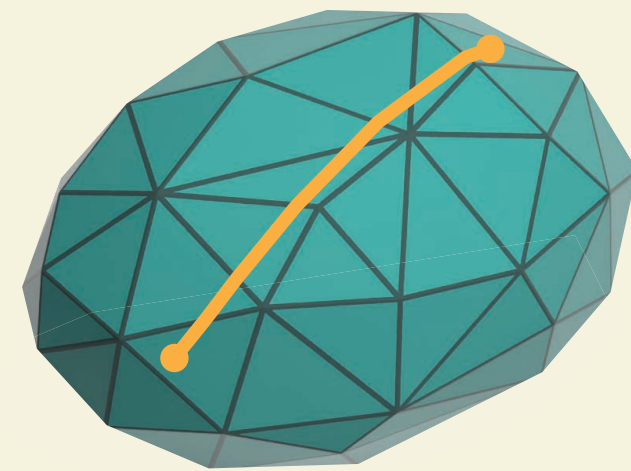
determines triangle in plane up to rigid motion



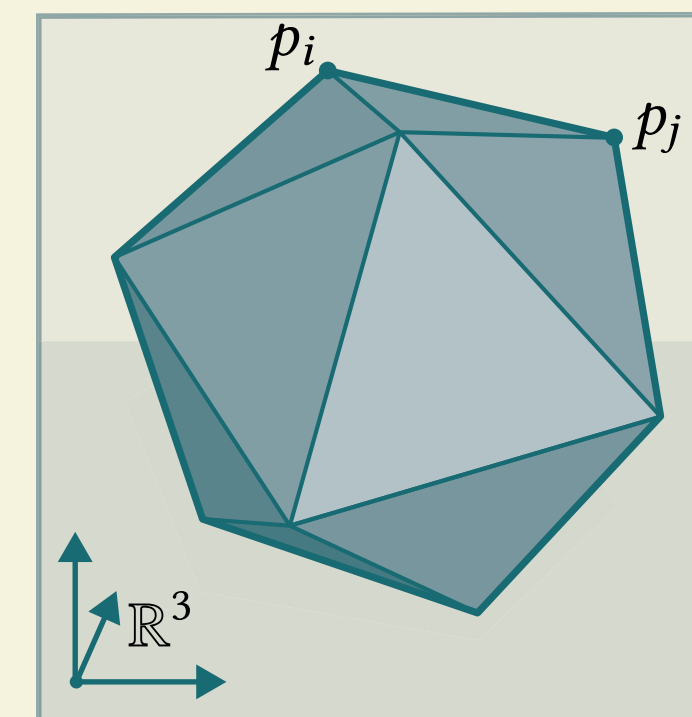
enough for many common geometry processing tasks

Compute:

- face areas
- corner angles
- shortest paths
- spectral calculations $Lu = \lambda u$

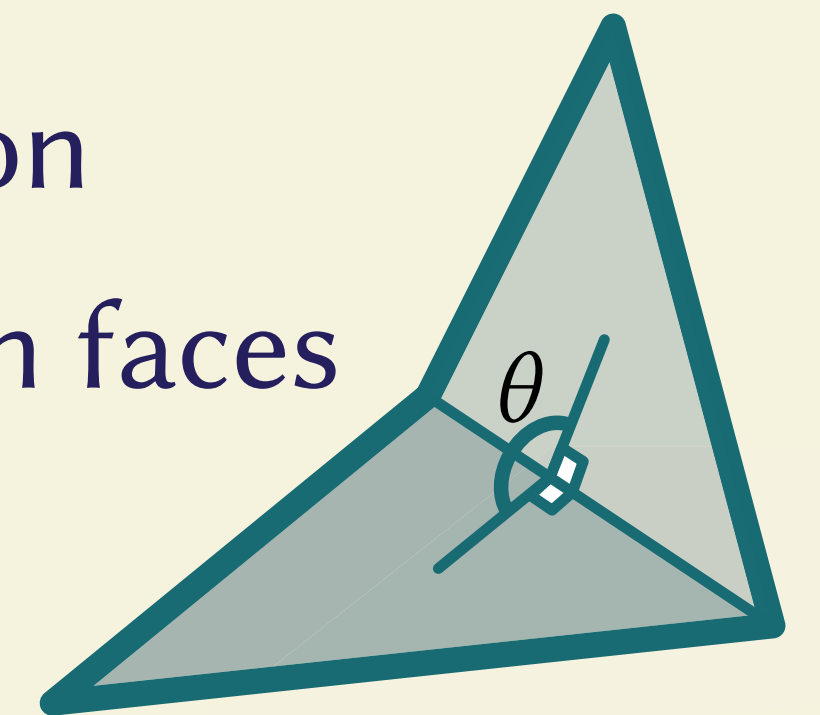


extrinsic

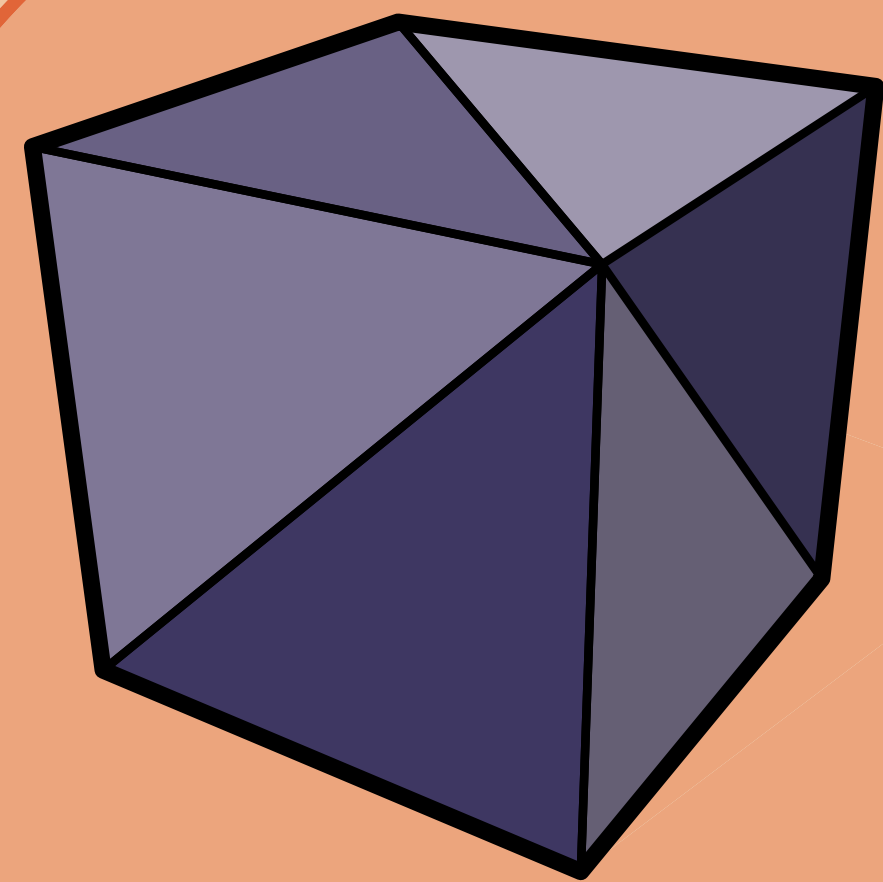


Compute:

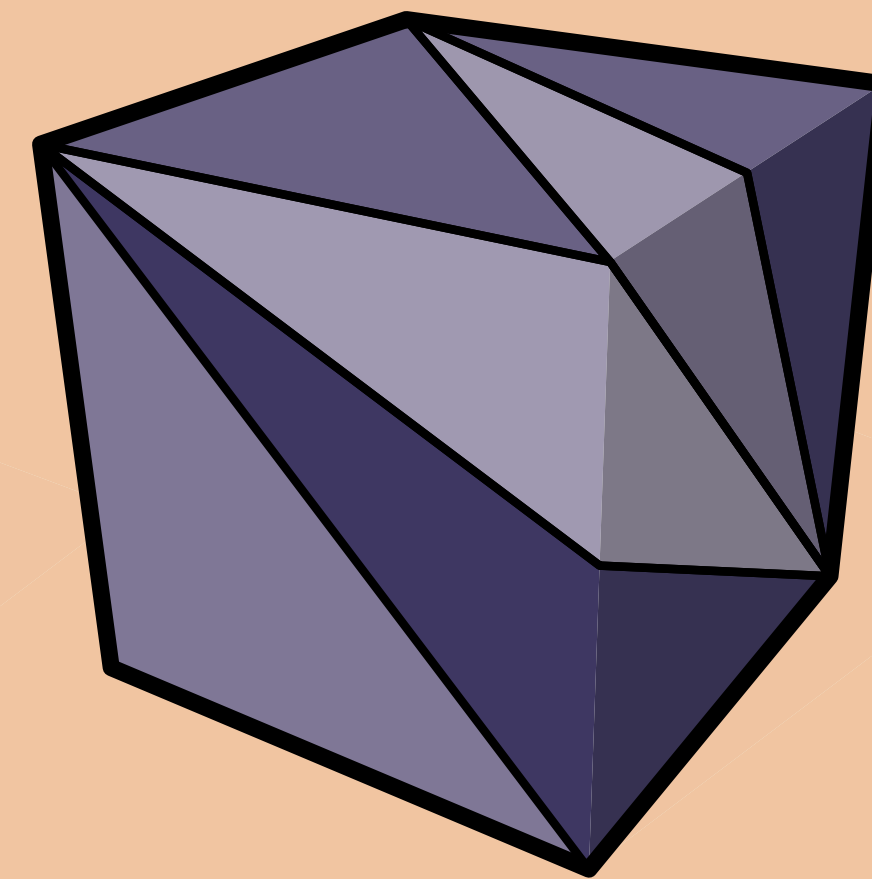
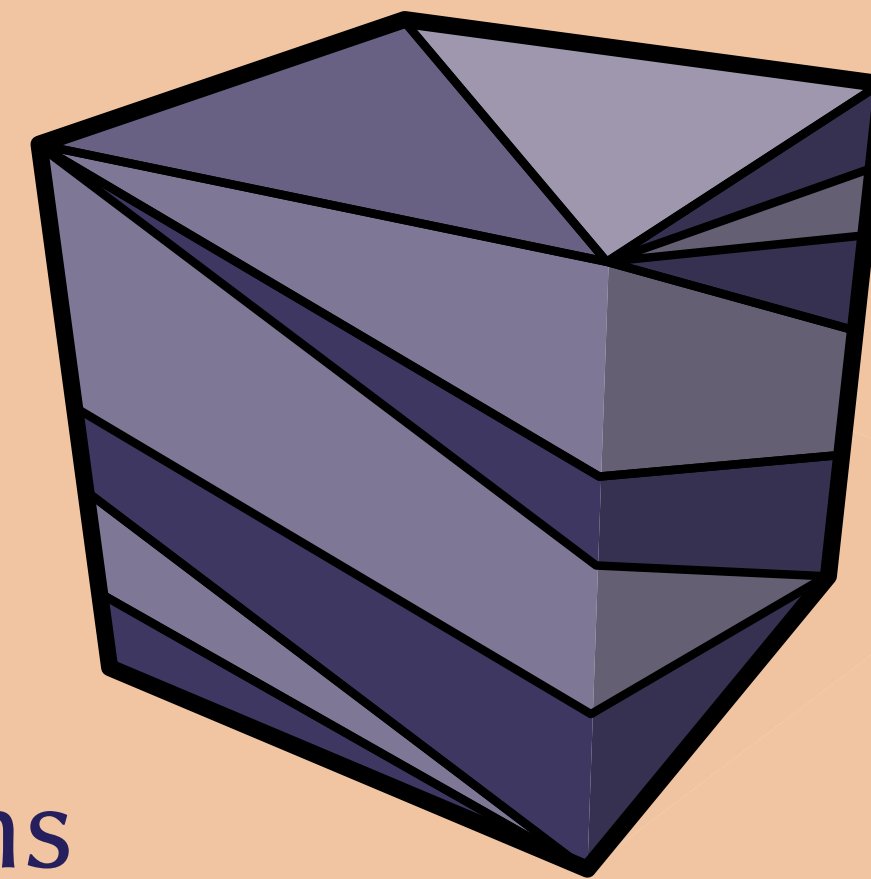
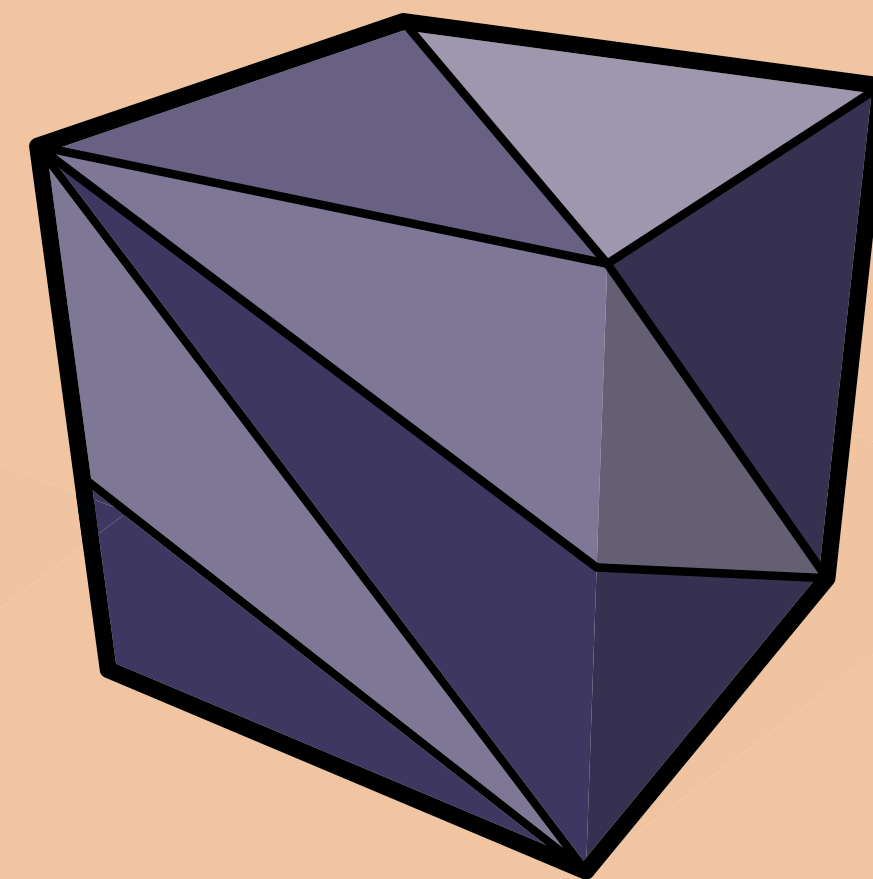
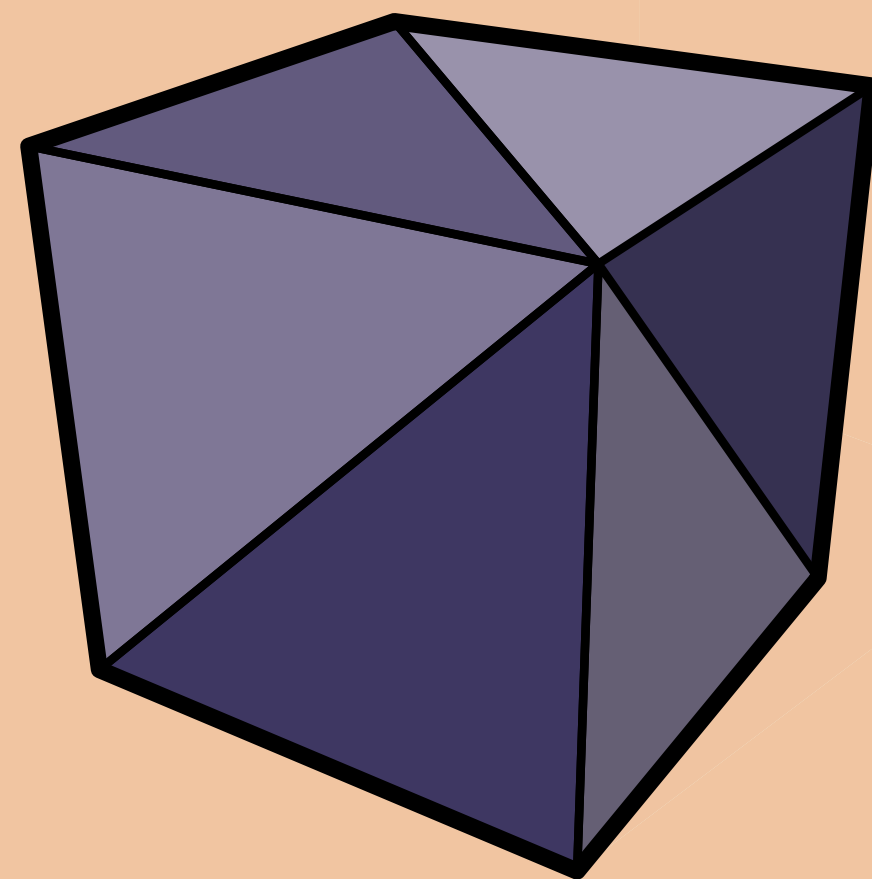
- any intrinsic information
- bending angles between faces



The space of intrinsic triangulations is large



extrinsic
triangulations



...

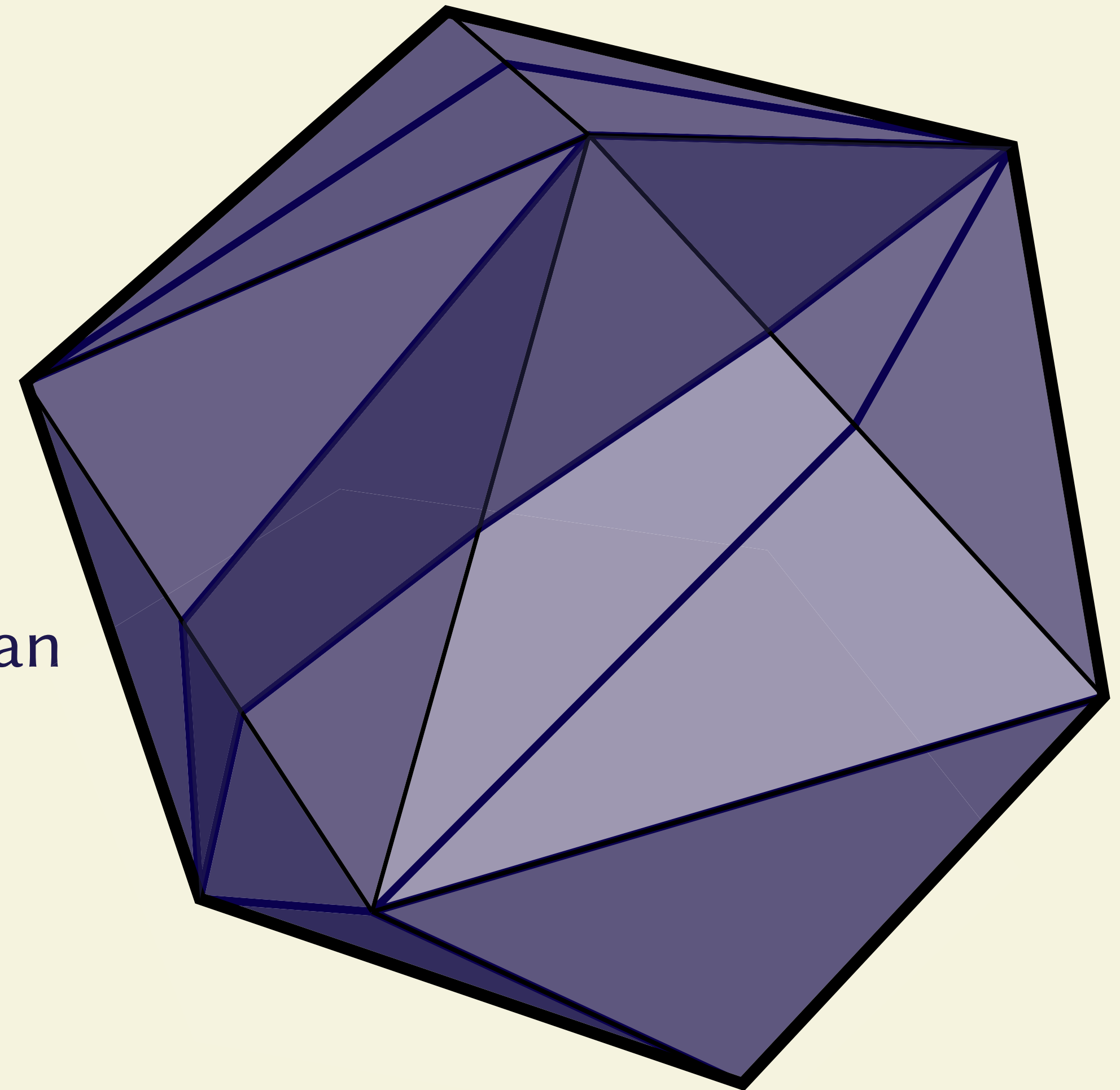
intrinsic
triangulations

Correspondence

Definition

A *correspondence* between two triangulations is a homeomorphism between them

- Usual case: intrinsic triangulation sitting on top of an extrinsic triangulation
- *i.e.* isometric or at least homeomorphic to some extrinsic triangulation

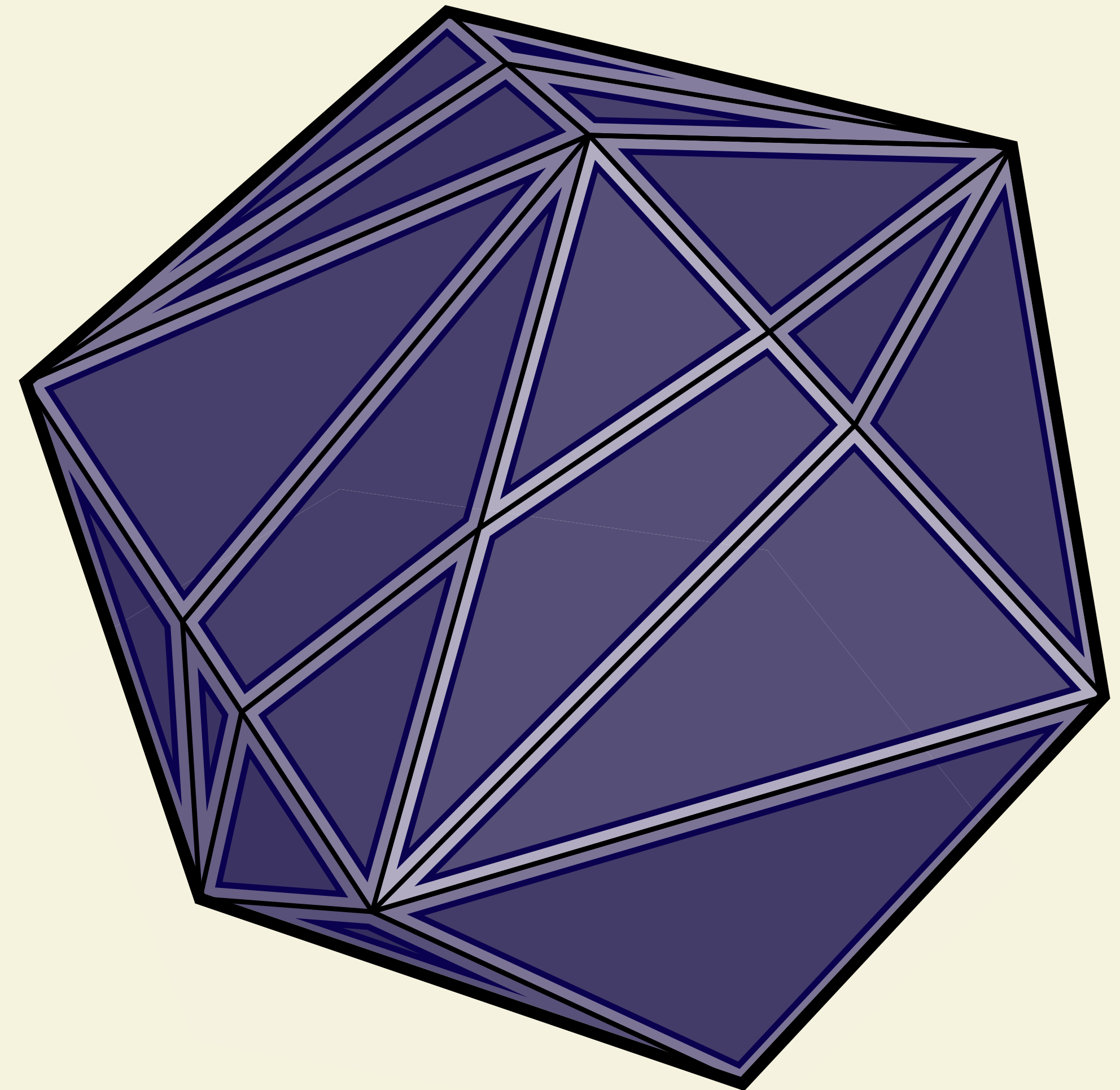


Common subdivision

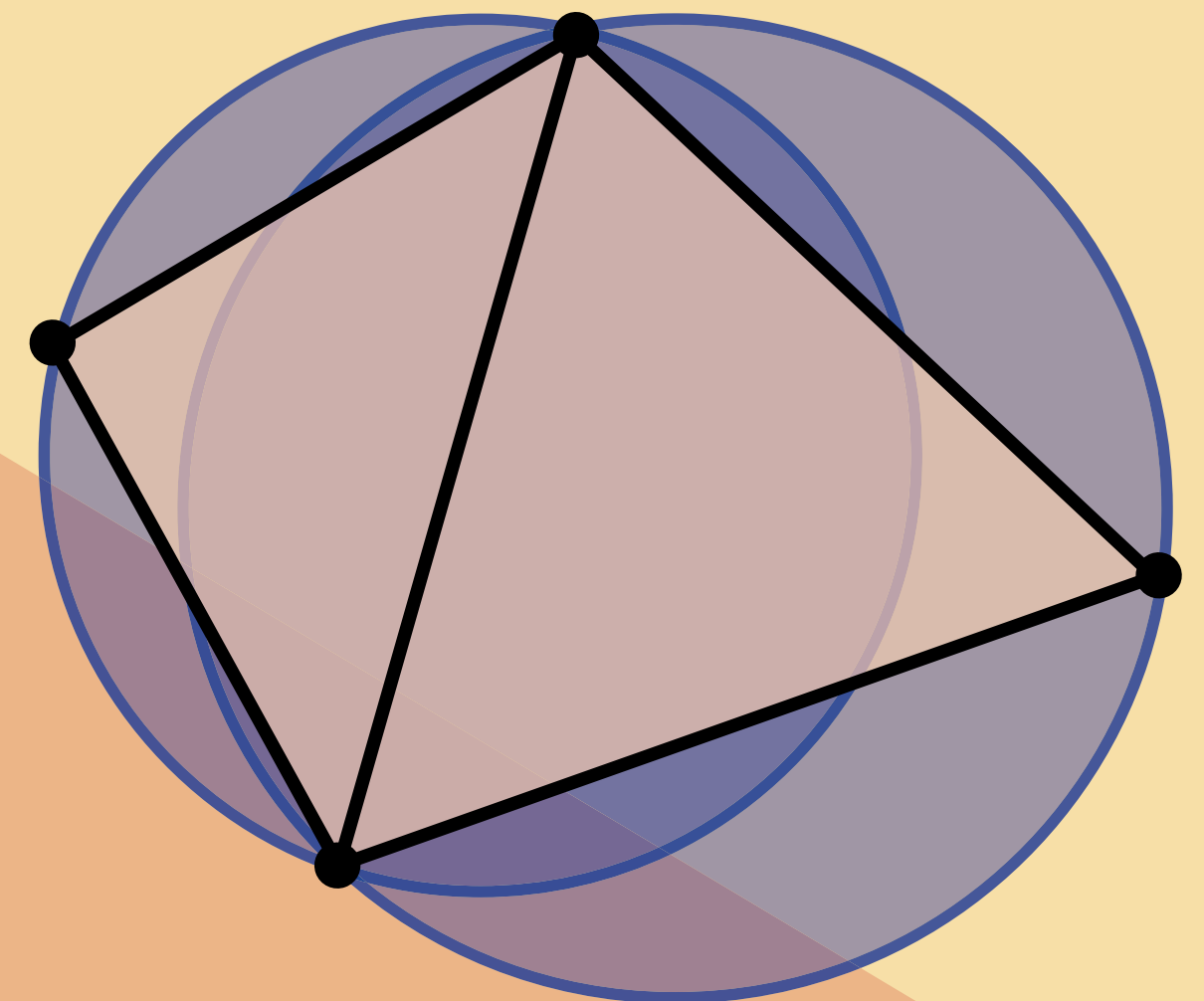
Definition

The *common subdivision* of two triangulations is the coarsest polygonal complex $\mathcal{S}$ such that all faces of either triangulation are unions of faces of $\mathcal{S}$

Intuitively, the result of cutting T_1 along edges of T_2



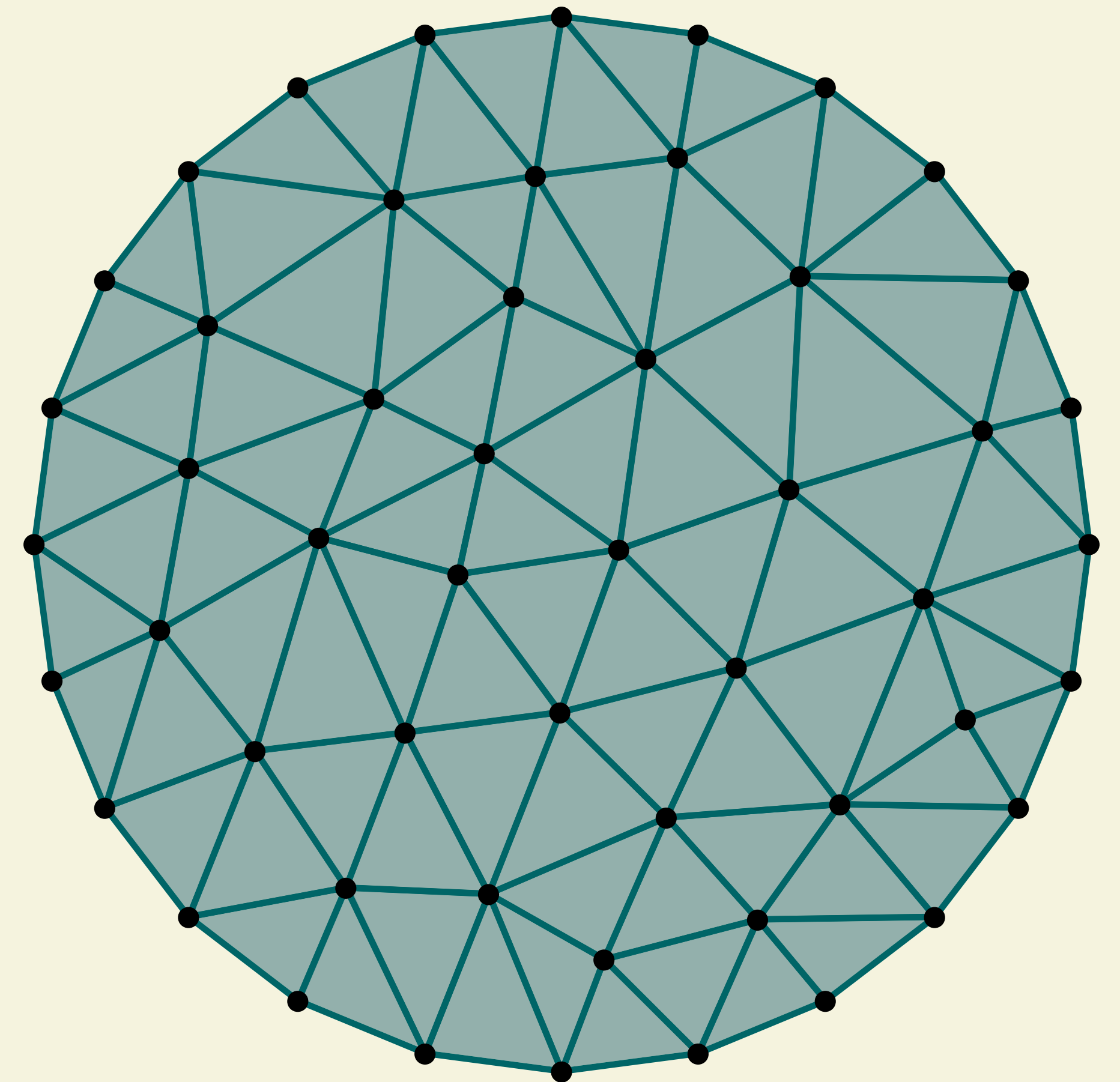
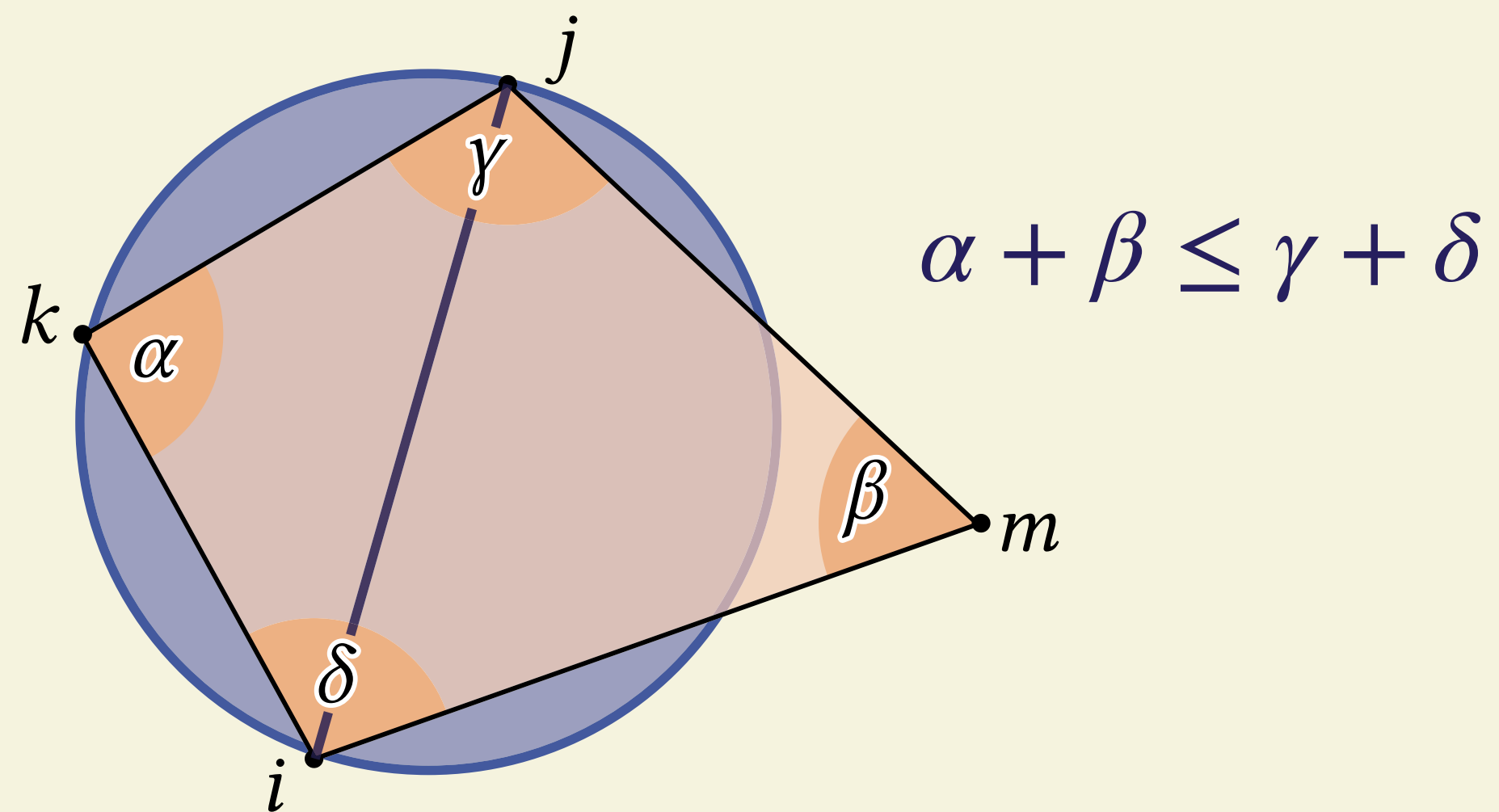
II. Delaunay Triangulations



[Sharp, G. & Crane. 2021. Geometry Processing with
Intrinsic Triangulations. *SIGGRAPH '21 Courses*]

Delaunay triangulations

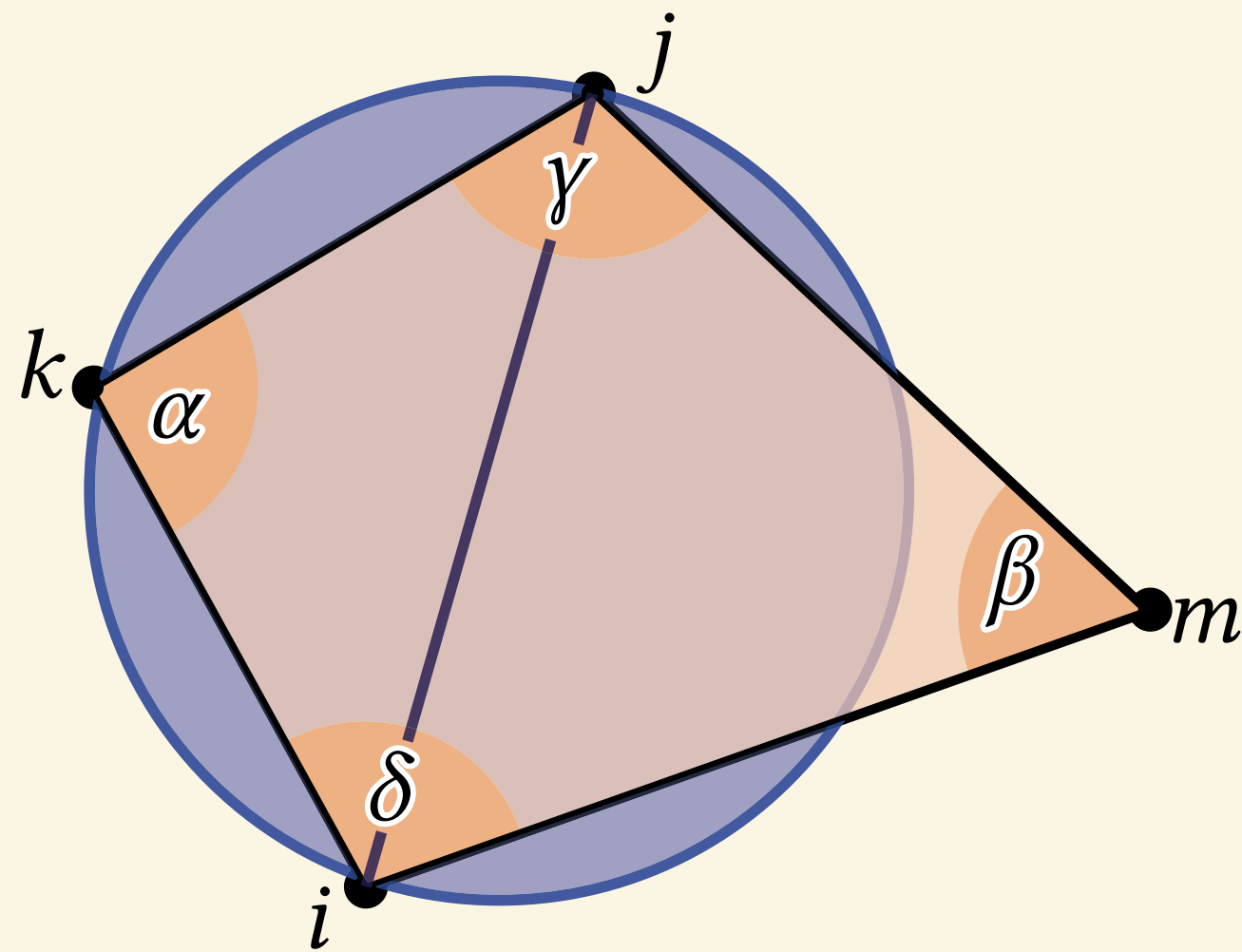
- Countless useful properties:
 - Essentially unique, maximize angles lexicographically, minimize spectrum lexicographically, smoothest interpolation, positive cotan weights...
- Characterized by *empty circumcircle condition*



Computing planar Delaunay triangulations

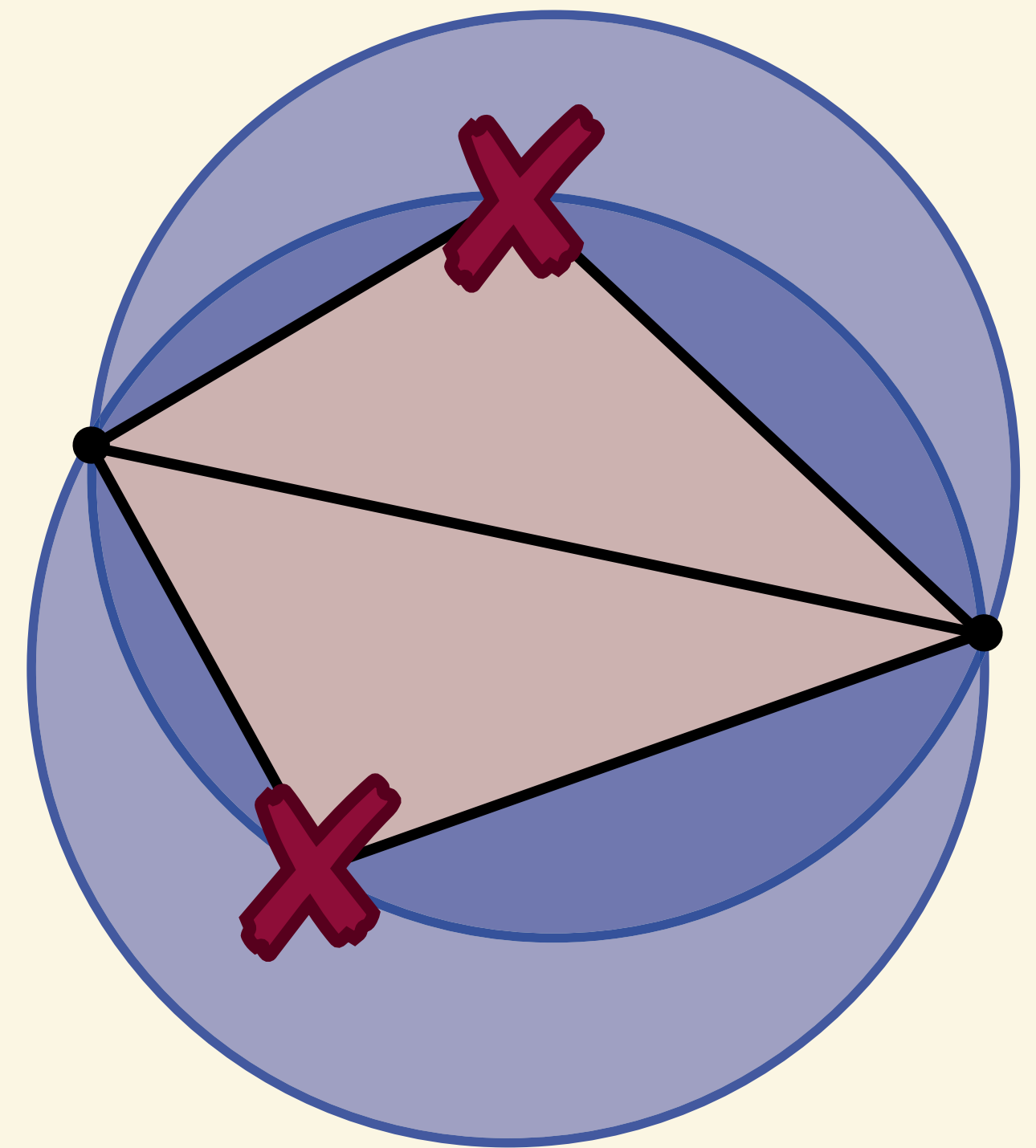
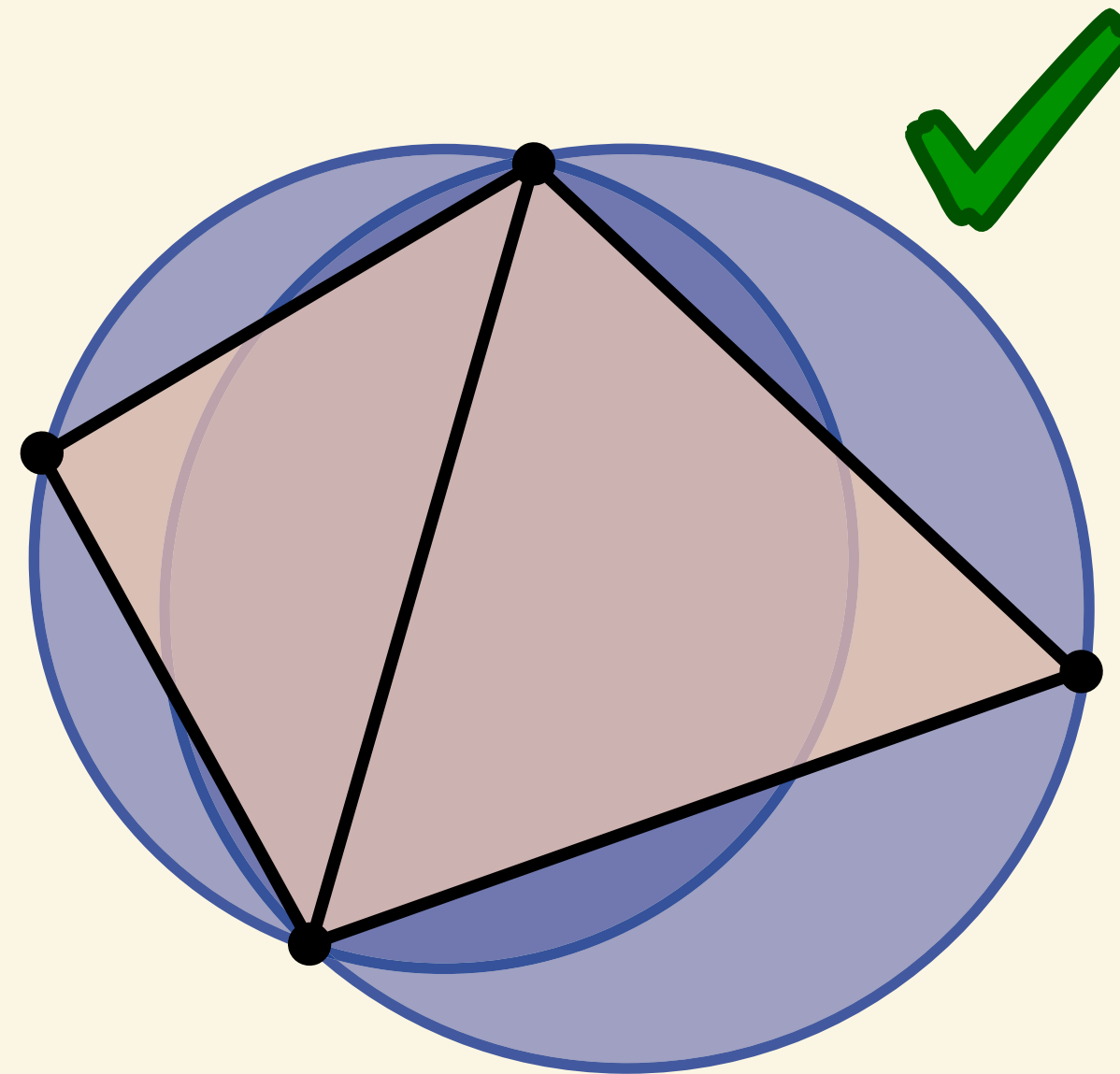
edge flips

empty circumcircle condition



$$\alpha + \beta \leq \gamma + \delta$$

(Assuming vertices do not all lie on a common circle. If they do, define a polygonal face)

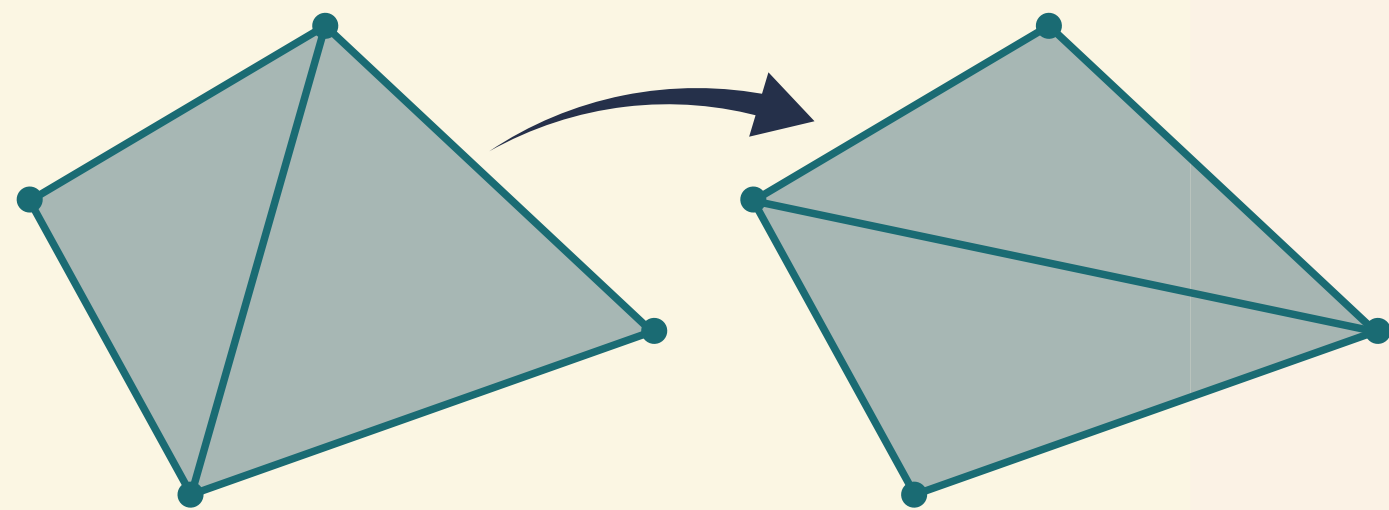


Computing planar Delaunay triangulations

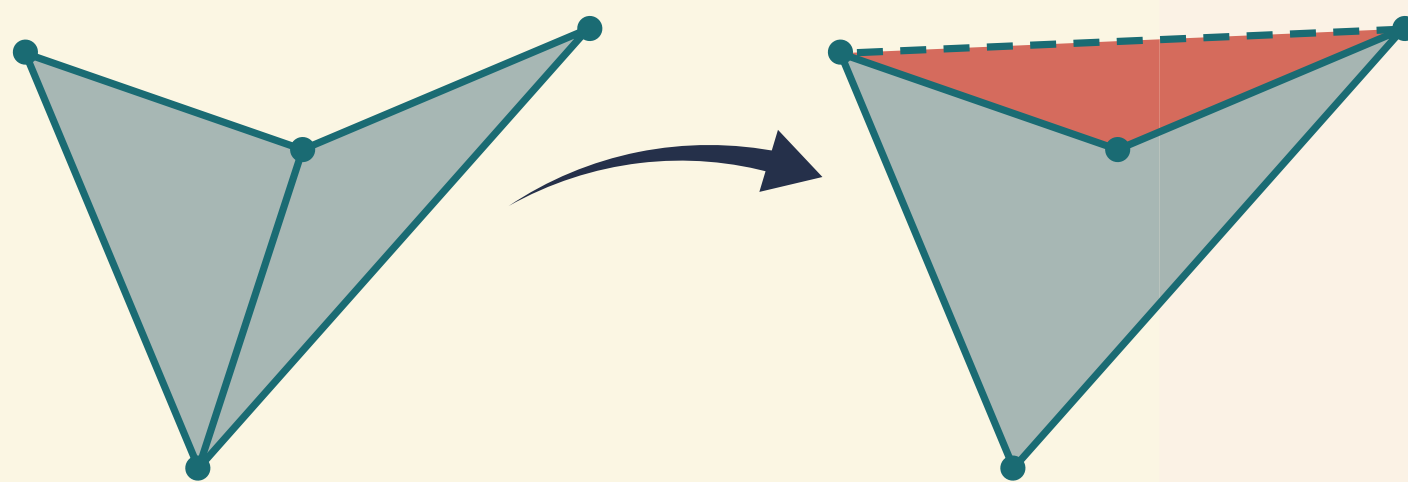
the flip algorithm [Lawson 1977]

Flipability:

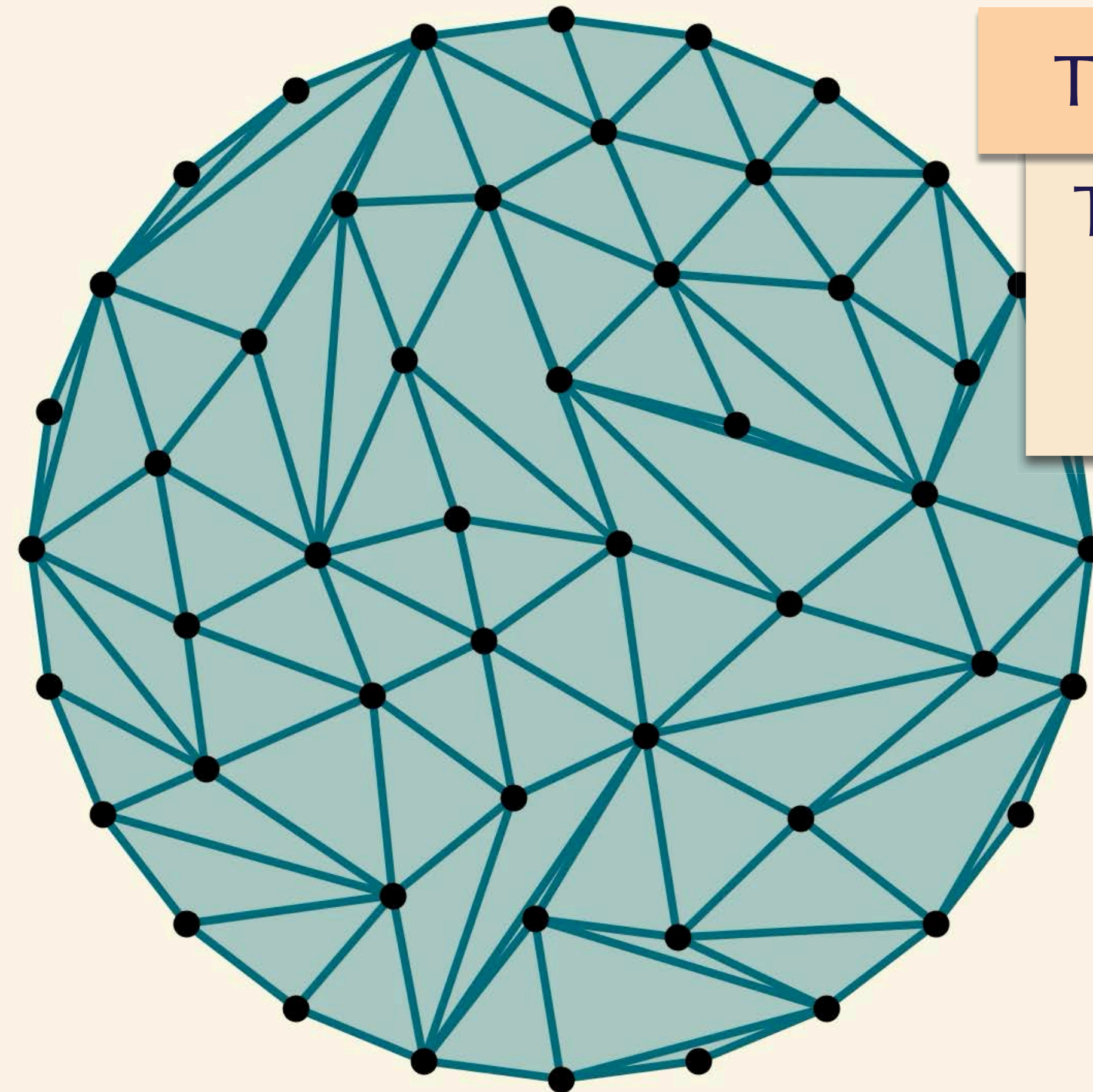
convex quad — always flipable



nonconvex quad — not flipable



but always Delaunay



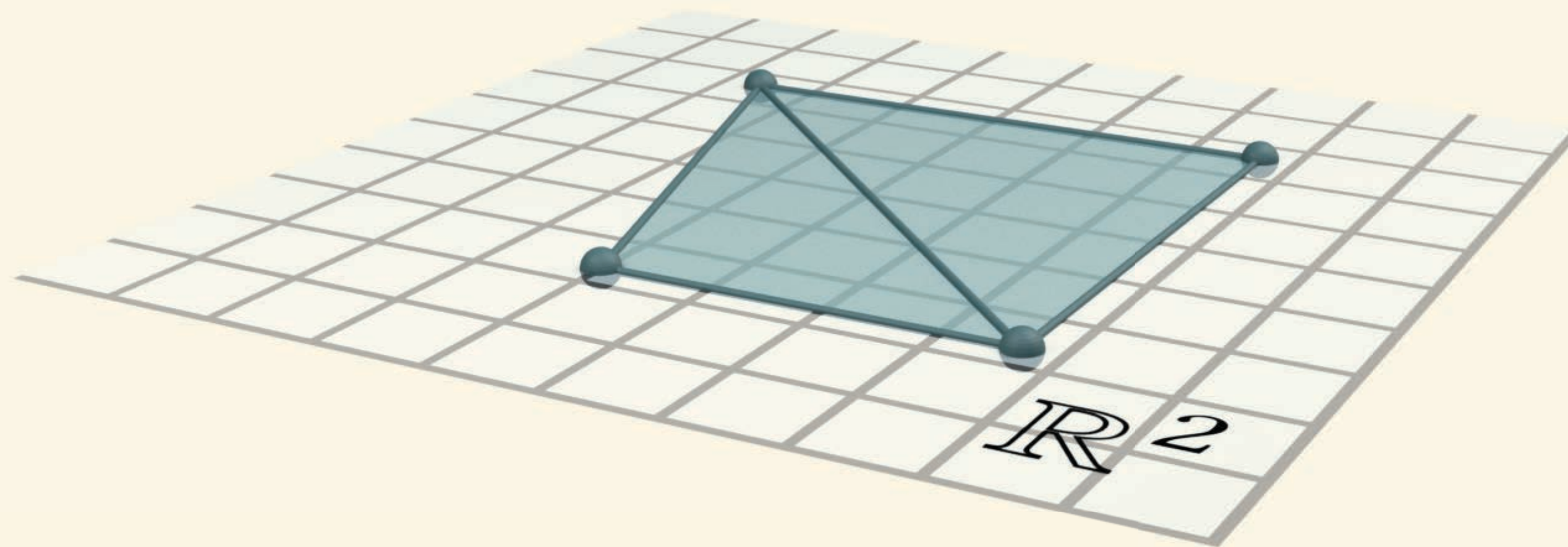
Theorem [Lawson 1972, 1977]

The flip algorithm computes the Delaunay triangulation in $O(|V|^2)$ flips

key lemma: each edge is flipped at most once

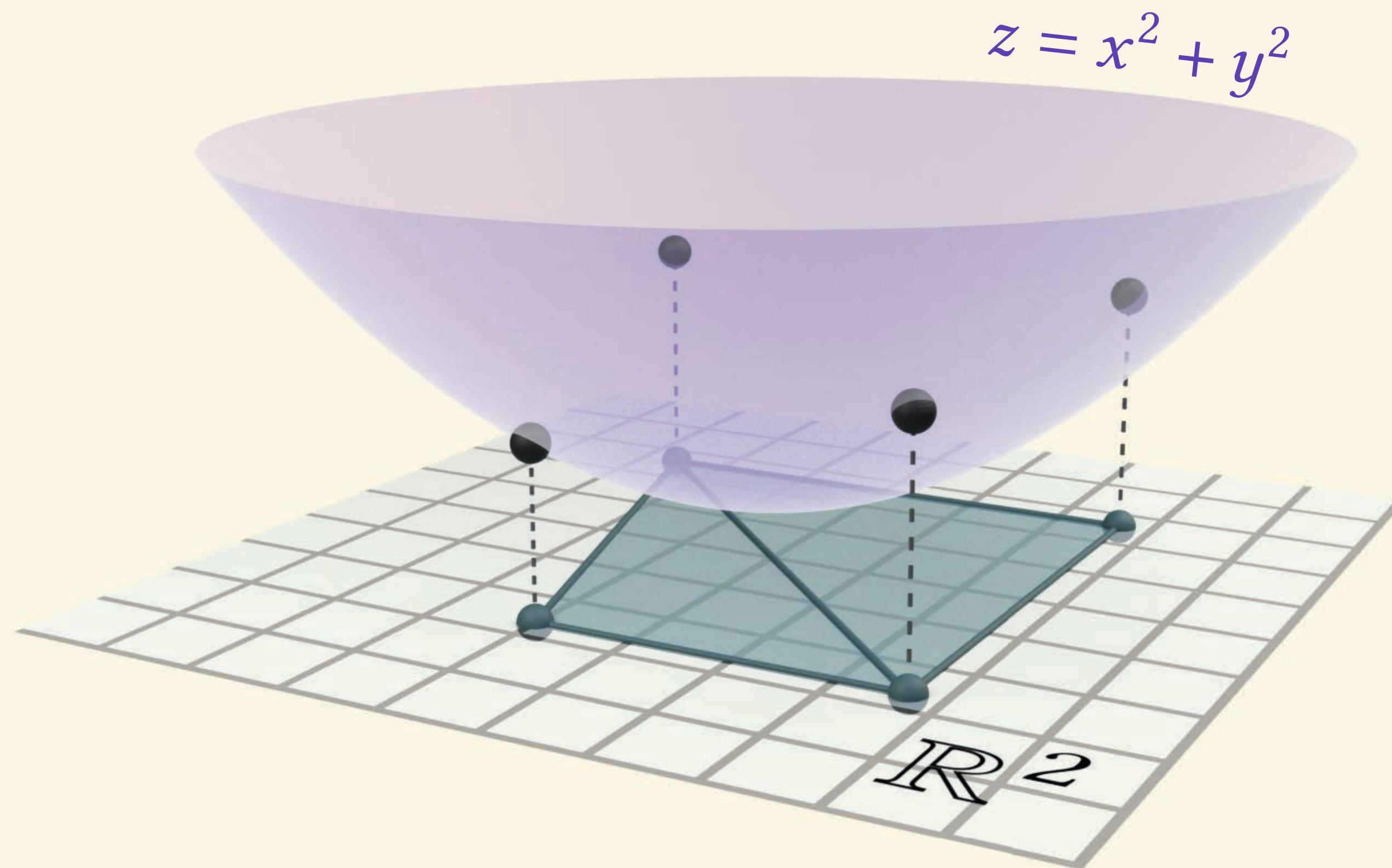
Computing planar Delaunay triangulations

the lifting picture



Computing planar Delaunay triangulations

the lifting picture



Computing planar Delaunay triangulations

the lifting picture

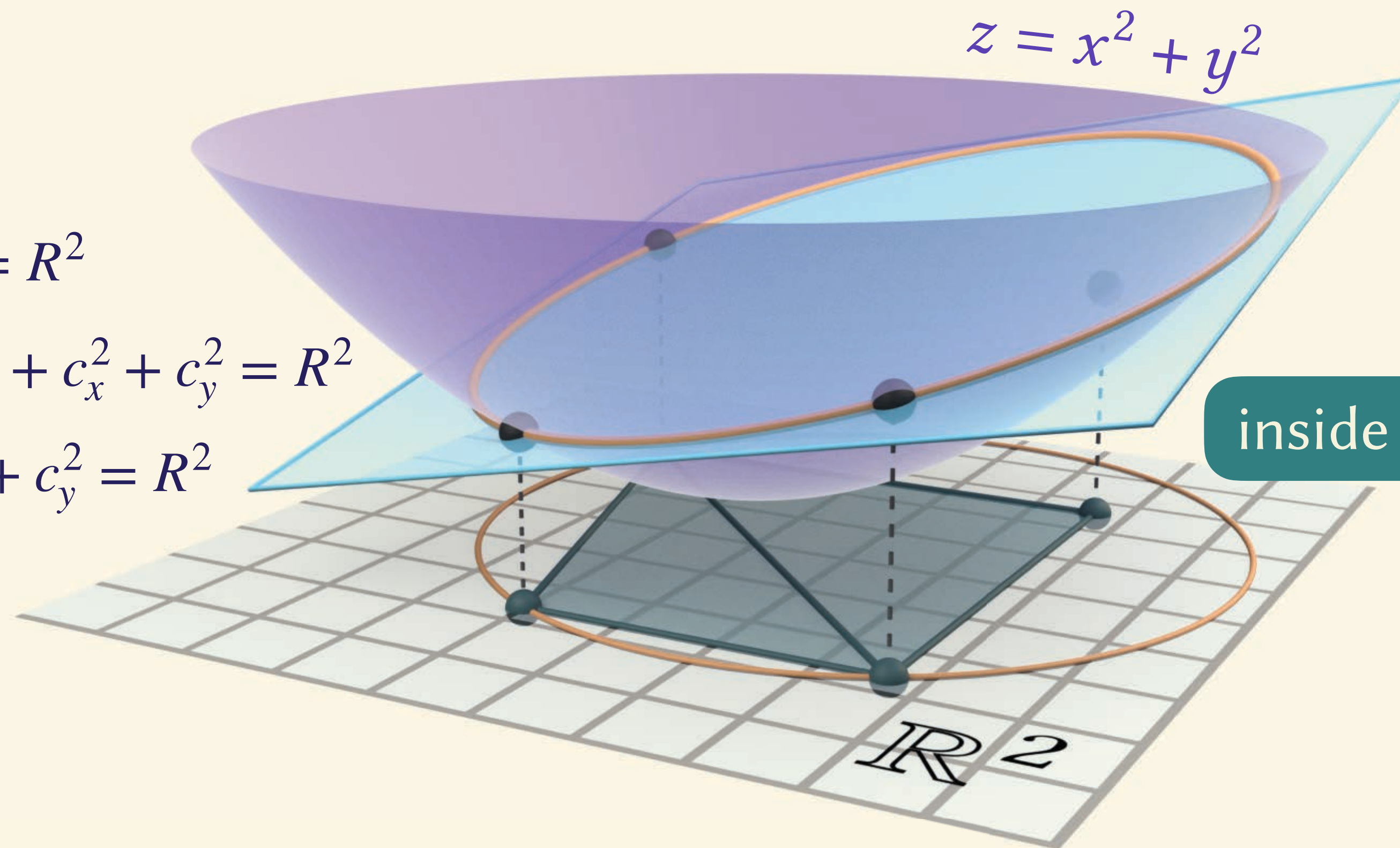
circle lifts to slice
by affine plane:

$$(x - c_x)^2 + (y - c_y)^2 = R^2$$

$$x^2 + y^2 - 2c_x x - 2c_y y + c_x^2 + c_y^2 = R^2$$

$$z - 2c_x x - 2c_y y + c_x^2 + c_y^2 = R^2$$

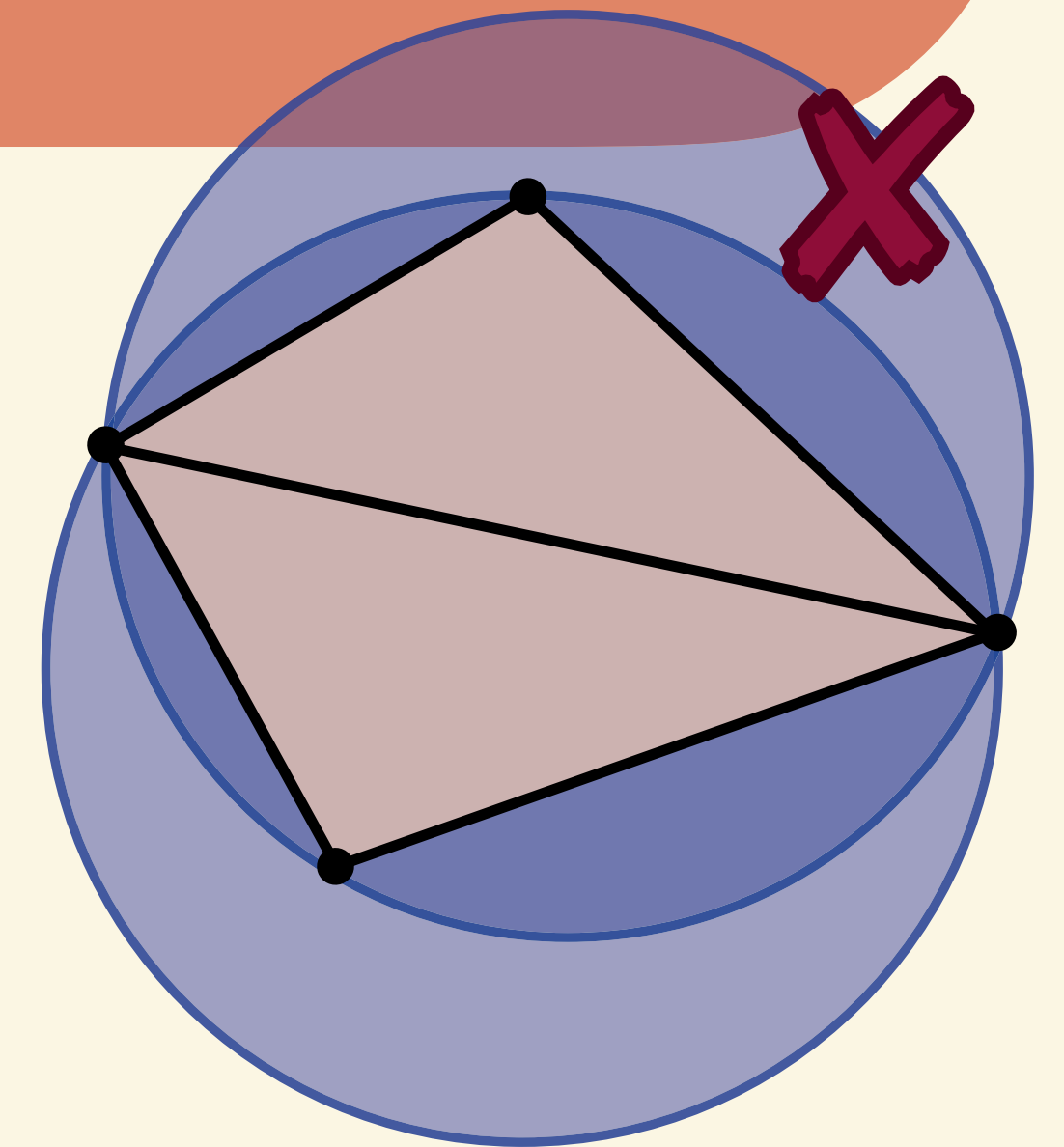
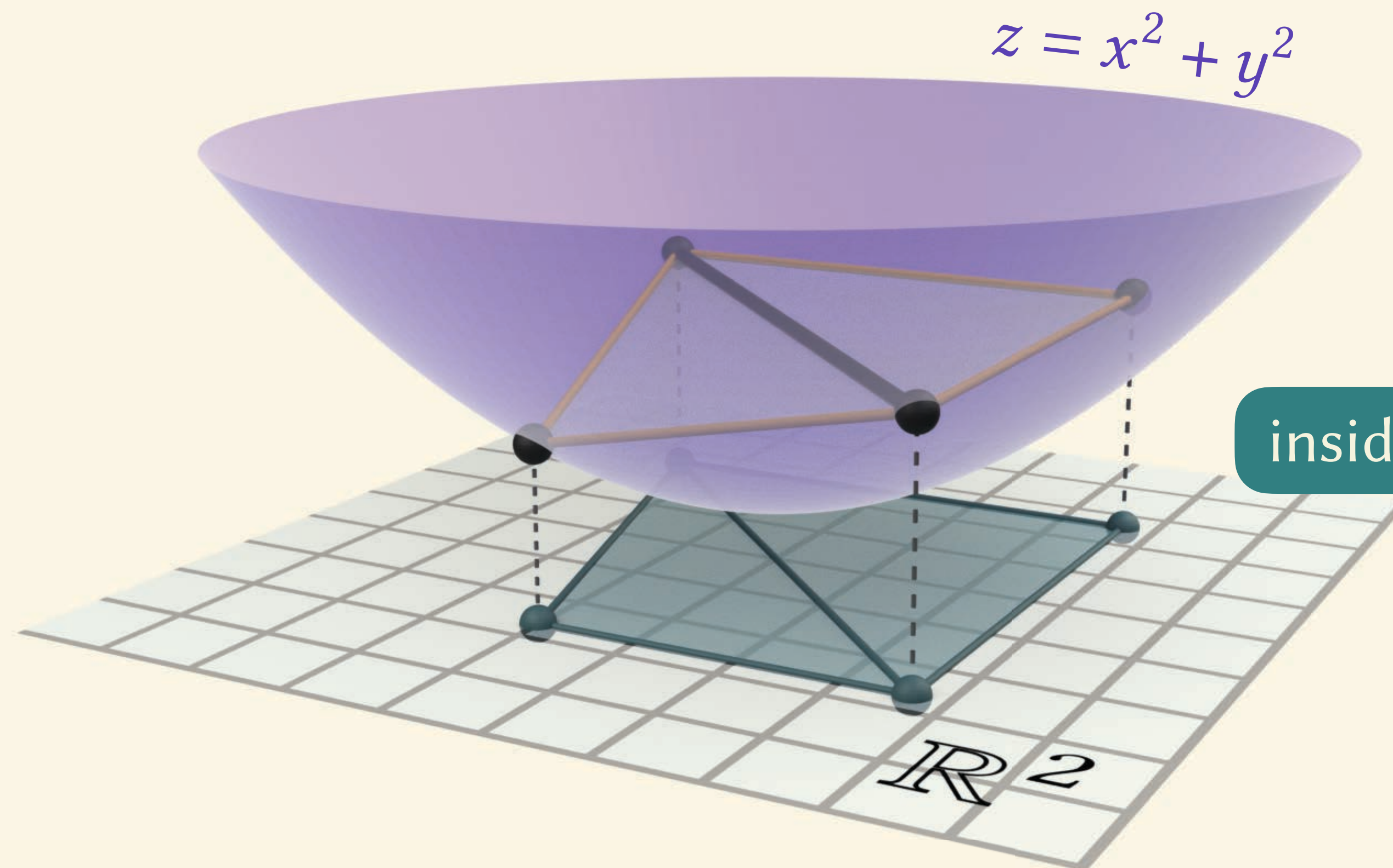
linear in x, y, z



inside circle $\iff$ underneath plane

Computing planar Delaunay triangulations

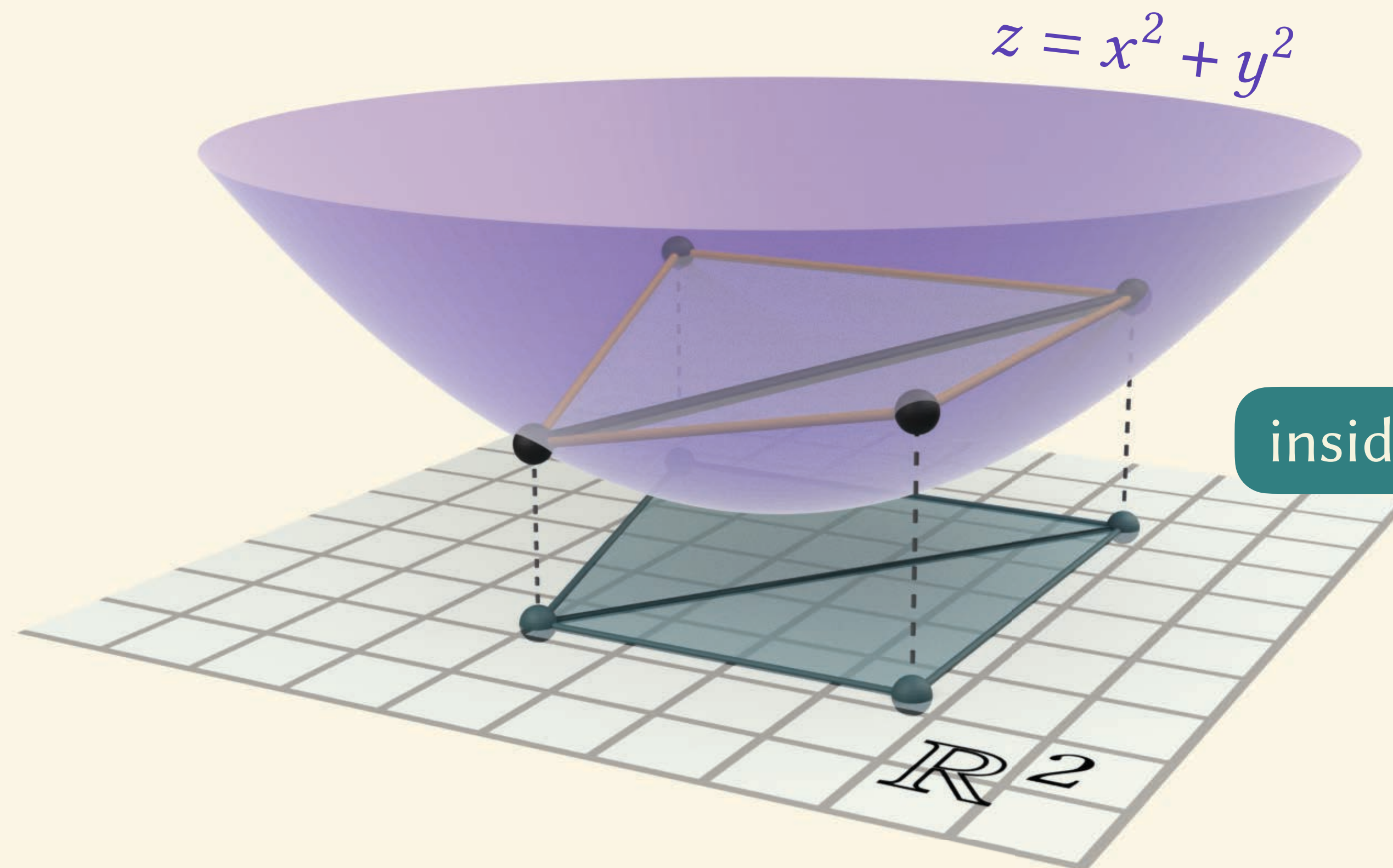
the lifting picture



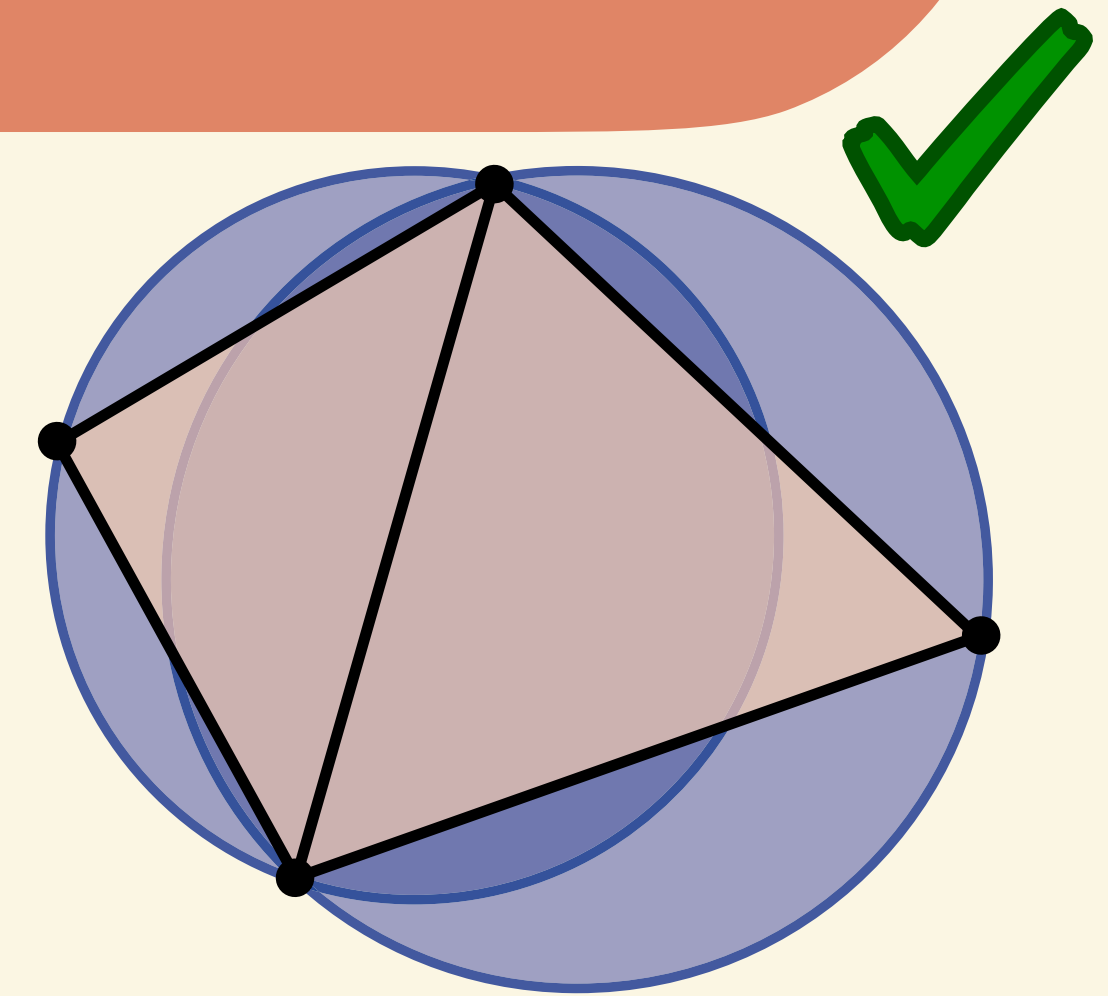
inside circle $\iff$ underneath plane

Computing planar Delaunay triangulations

the lifting picture



inside circle $\iff$ underneath plane



Computing planar Delaunay triangulations

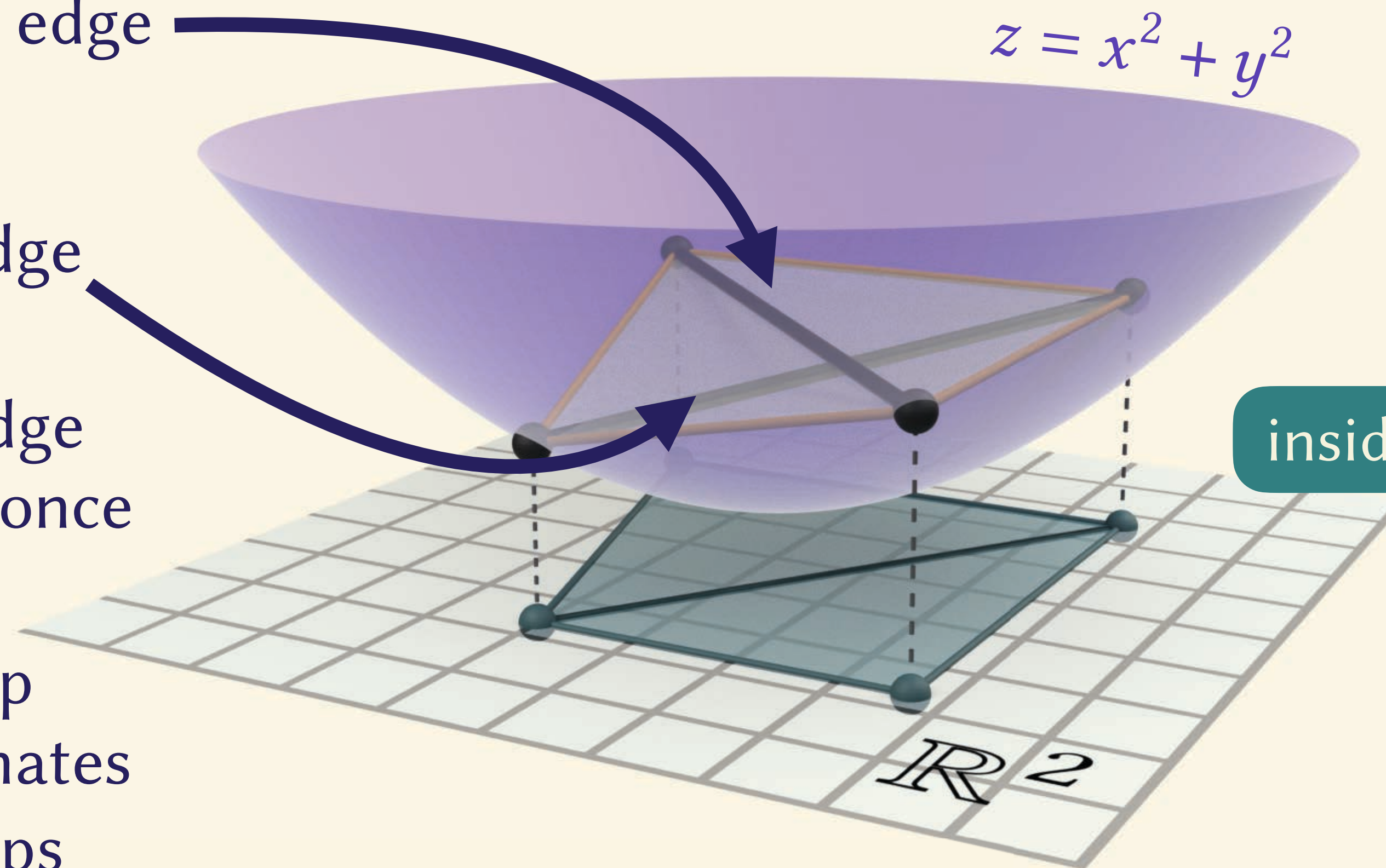
the lifting picture

replaced a higher edge

with a lower edge

and so every edge
appears at most once

and so the flip
algorithm terminates
in $O(|V|^2)$ flips



inside circle $\iff$ underneath plane

Computing planar Delaunay triangulations

flipping can take quadratic time

Theorem [Fortune 1993]

There exist triangulations on which Lawson's algorithm takes $\Omega(|V|^2)$ flips.

and so the flip
algorithm terminates
in $O(|V|^2)$ flips

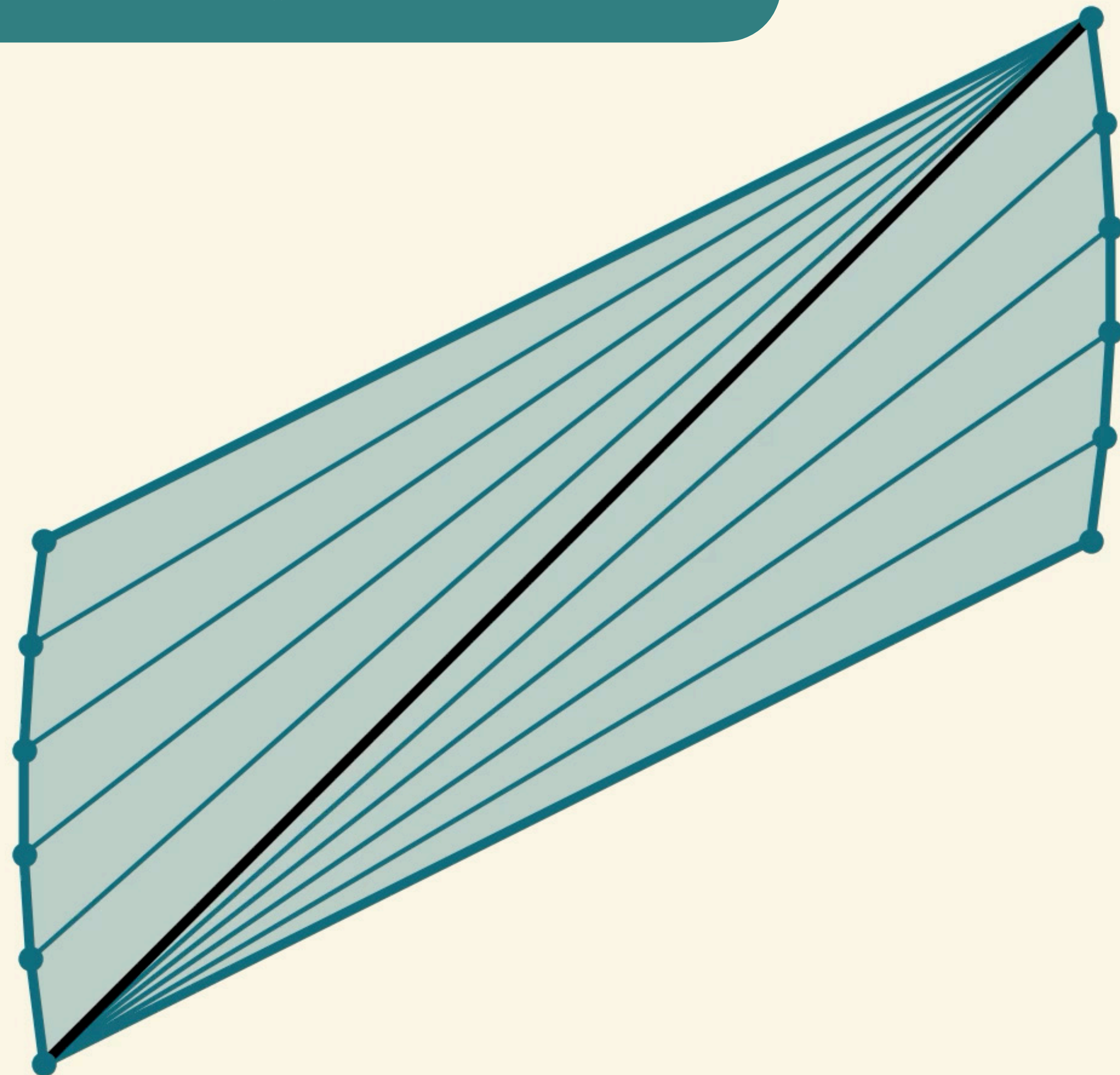
Computing planar Delaunay triangulations

flipping can take quadratic time

Theorem [Fortune 1993]

There exist triangulations on which Lawson's algorithm takes $\Omega(|V|^2)$ flips.

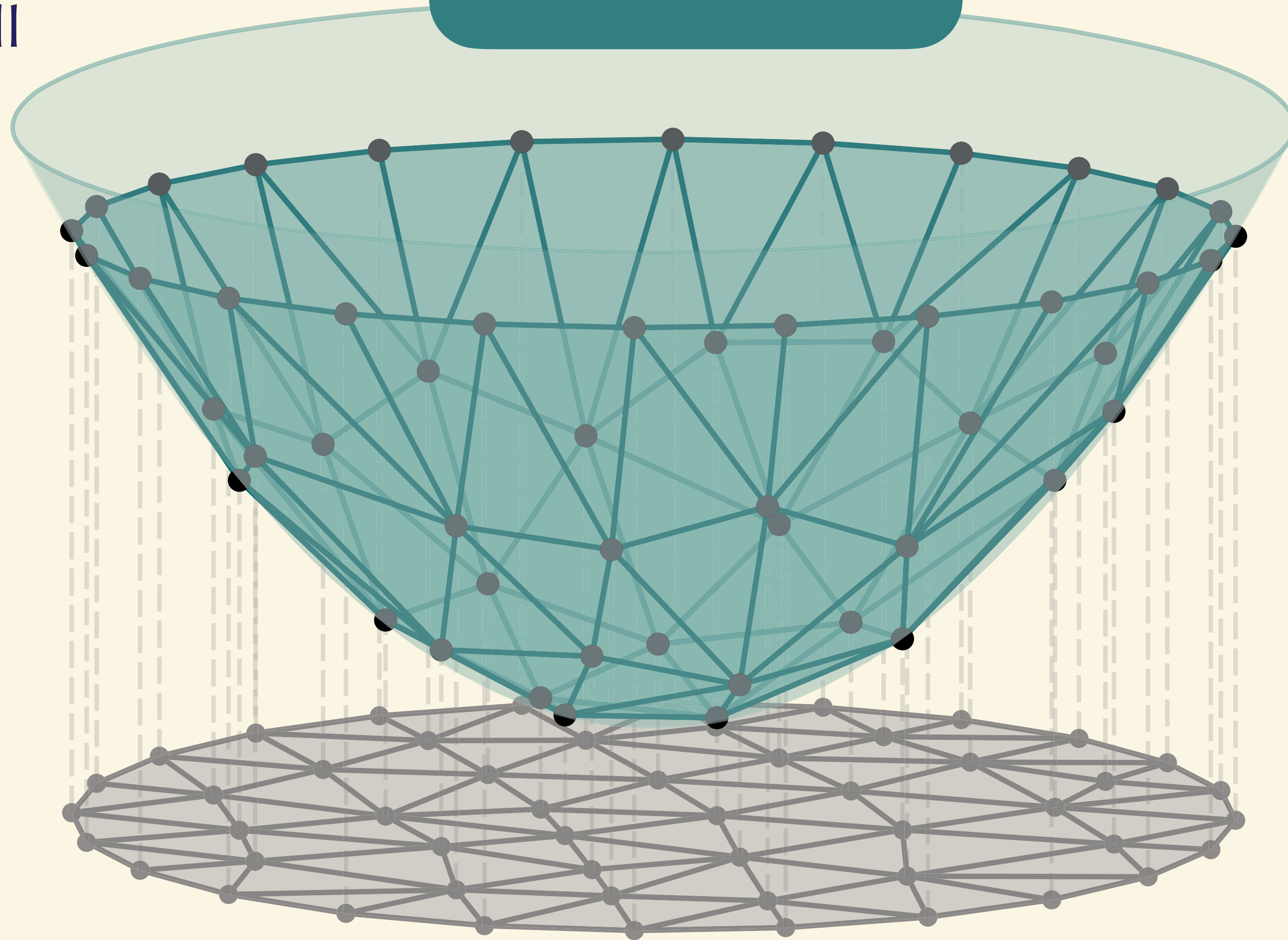
and so the flip algorithm terminates in $O(|V|^2)$ flips



Computing planar Delaunay triangulations

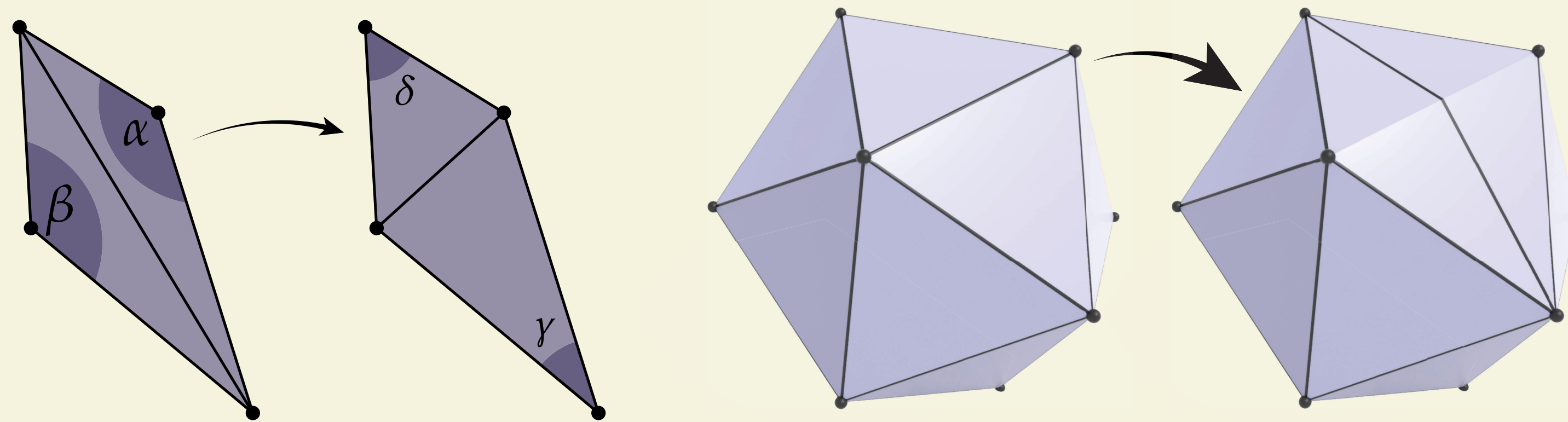
via convex hulls

compute convex hull
in time
 $O(|V|\log|V|)$

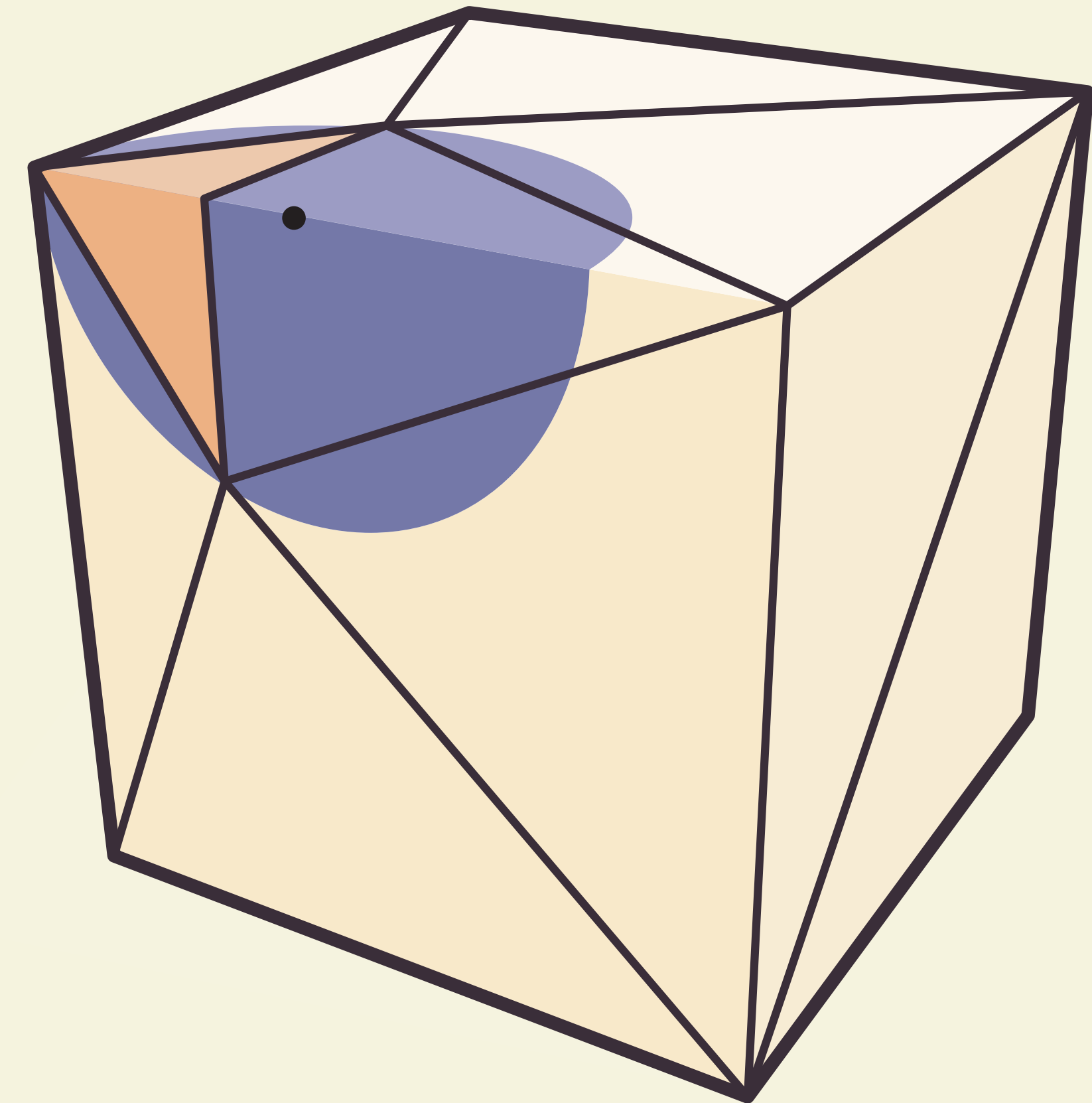


Intrinsic Delaunay triangulations

- [Indermitte, Liebling, Troyanov & Clemençon 2001, Bobenko & Springborn 2007]: empty intrinsic circumcircles
- Easy to compute by flip algorithm:
 - Flip any non-Delaunay edge until none remain



Maintains most important properties
[Sharp, G. & Crane 2021; §4.1.1]

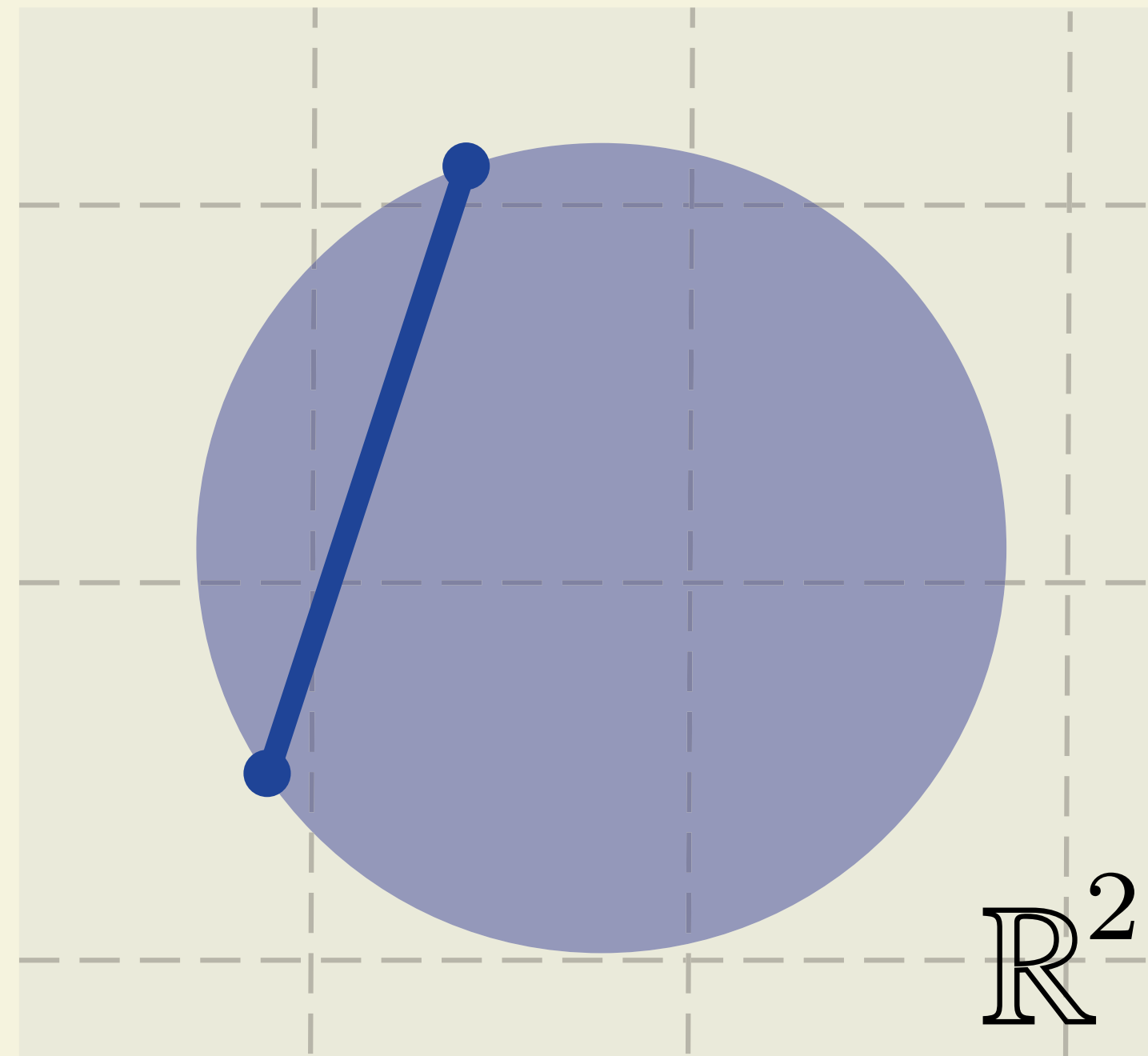


Properties of intrinsic Delaunay triangulations

uniqueness

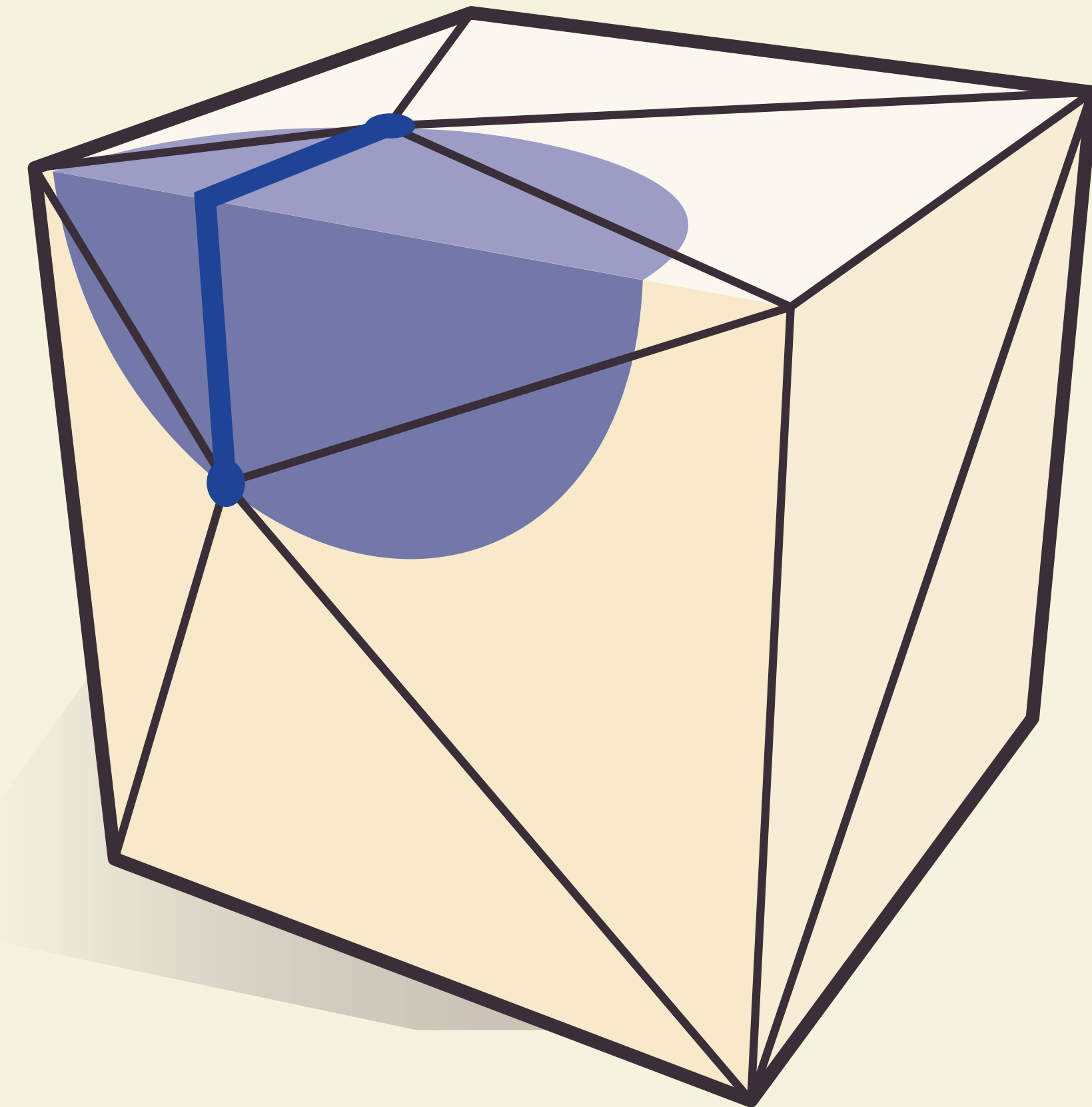
[Bobenko & Springborn 2007]

each edge sits in an empty disk



(in fact, all such edges are Delaunay)

two such edges can
only intersect at their
endpoints $\square$



Properties of intrinsic Delaunay triangulations

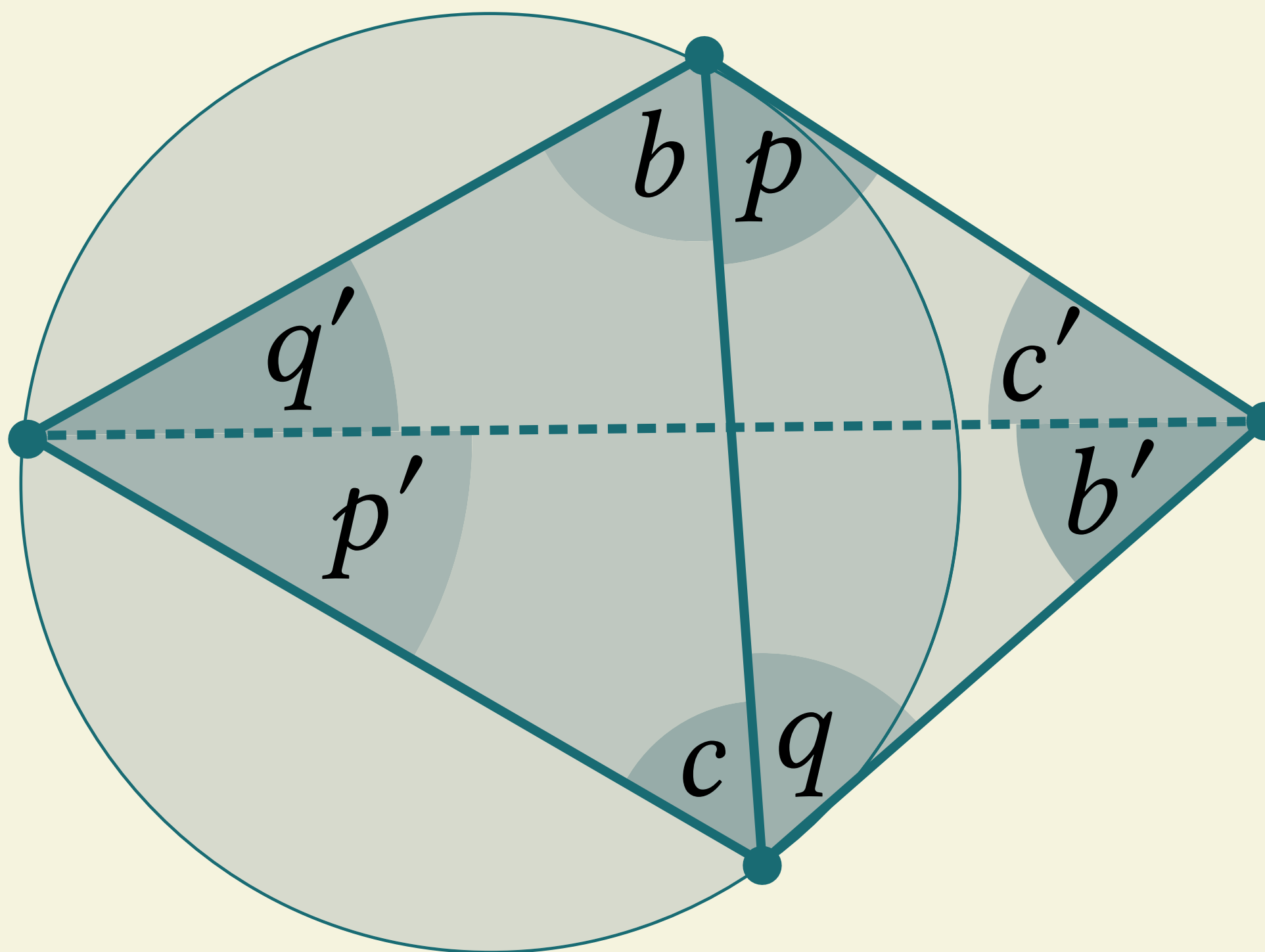
maximizes corner angles lexicographically

Theorem [Sibson 1978]

Let T_D be the Delaunay triangulation of a vertex set in general position (so no four vertices lie in a common circle), and let θ_{T_D} be the sorted list of corner angles of all triangles in T_D . For any other triangulation T , we have

$$\theta_{T_D} < \theta_T$$

under the lexicographic ordering.



each primed angle
is smaller than
unprimed angle

each Delaunay
flip increases
lexicographic rank $\square$

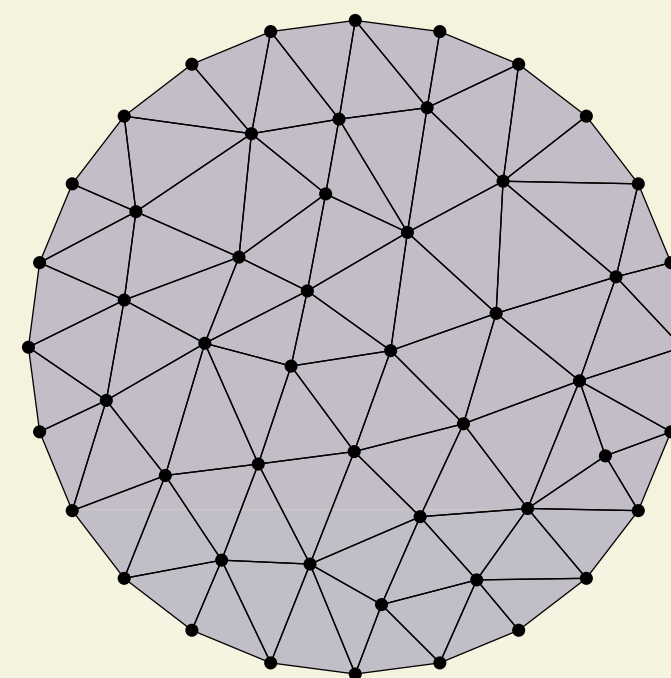
Properties of intrinsic Delaunay triangulations

smoothest piecewise-linear interpolation

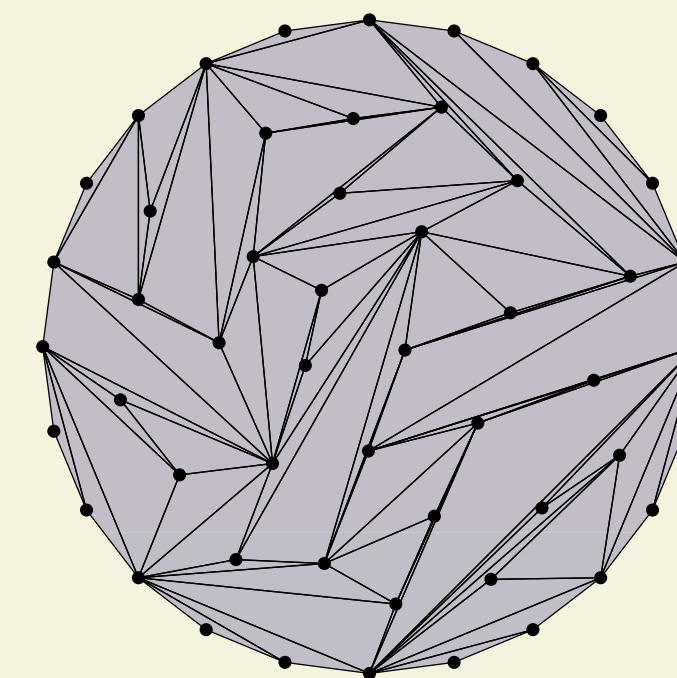
Theorem [Rippa 1990]

Fix a vertex set V in a domain S and any function $f: V \rightarrow \mathbb{R}$. For any triangulation T of the domain, let $f_T: S \rightarrow \mathbb{R}$ be the piecewise-linear interpolation of f . Then the minimum *Dirchlet energy* among all possible triangulations is realized by the Delaunay triangulation T_D :

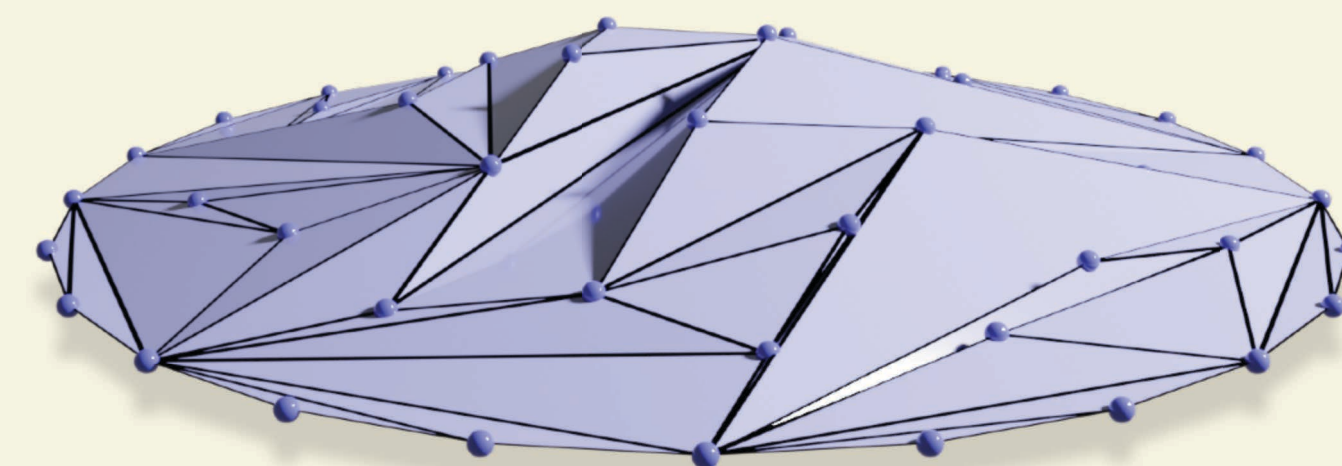
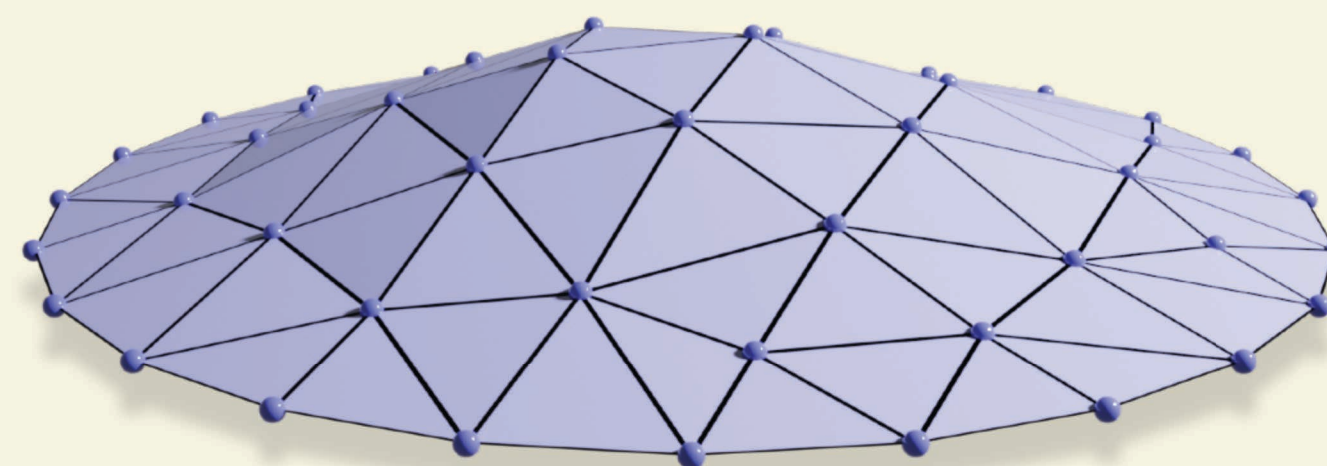
$$\int_S \|\nabla f_{T_D}\|^2 dx \leq \int_S \|\nabla f_T\|^2 dx \quad \forall T$$



Delaunay
triangulation



non-Delaunay
triangulation



a Delaunay flip can never increase this integral $\square$

Properties of intrinsic Delaunay triangulations

minimizes length of minimum spanning tree

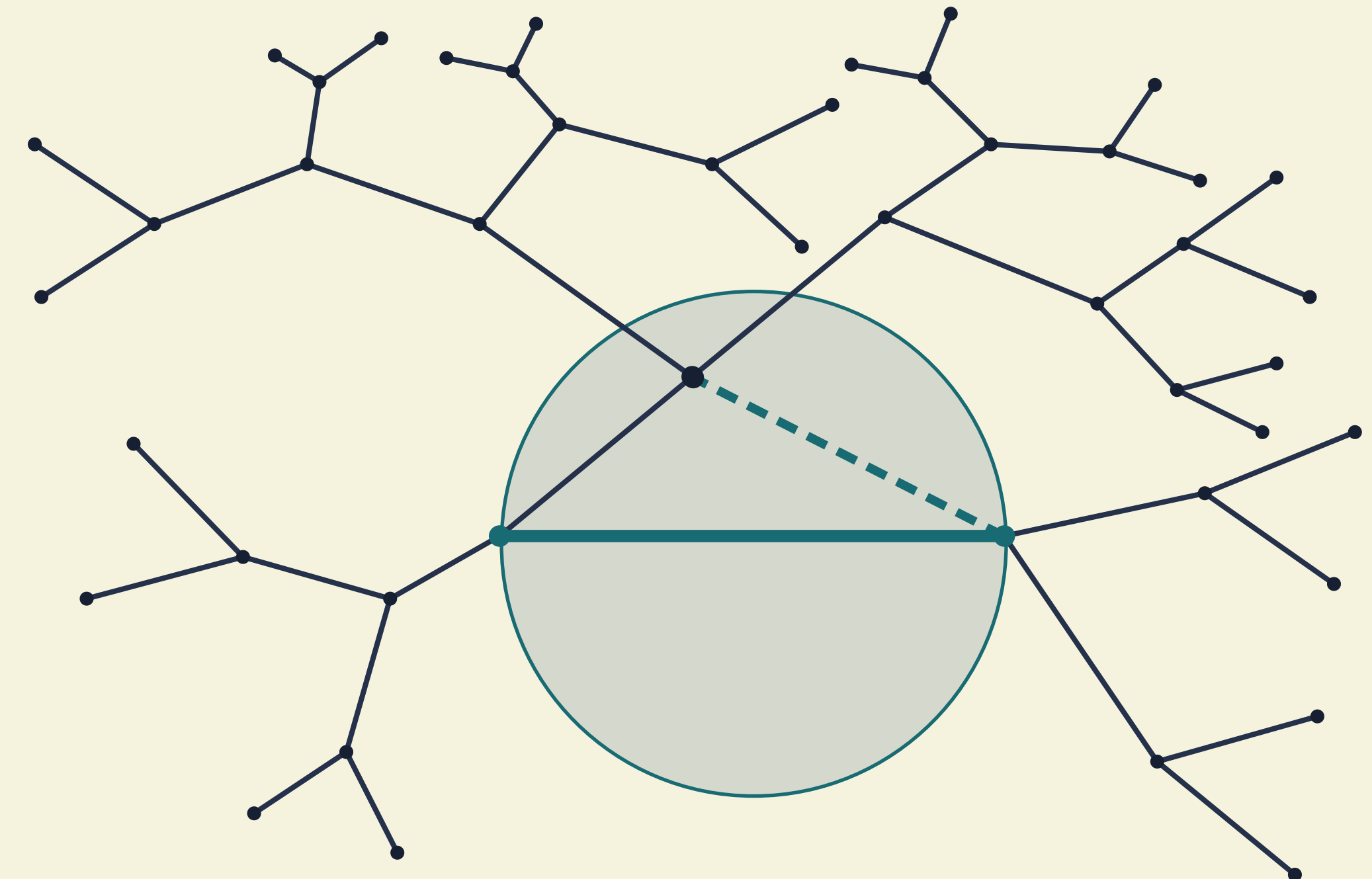
Theorem [Toussaint 1980; O'Rourke 1998]

The length of the minimum spanning tree in any triangulation T is at least the length of the minimum spanning tree in the Delaunay triangulation T_D .

Let M be the shortest such MST

Every edge of M is in T_D :

suppose for contradiction that one is not



contradicting the minimality of our initial tree $\square$

Properties of intrinsic Delaunay triangulations

... and more

Theorem [Xia 2013]

The Delaunay triangulation is a 2-spanner: the graph distance between any two vertices is at most twice the geodesic distance.

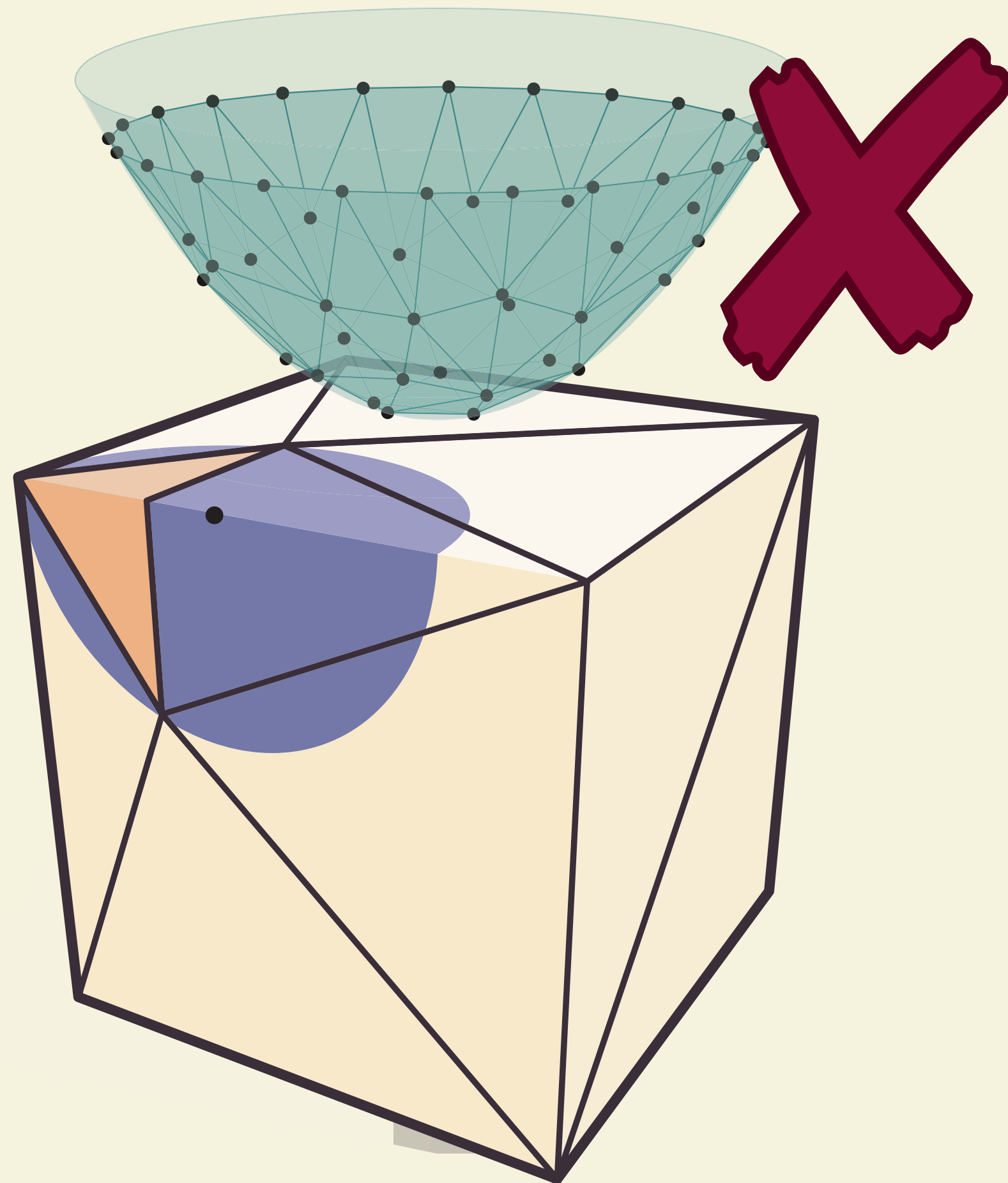
Theorem [Chen et al. 2010]

The eigenvalues of the Laplacian are minimized on a Delaunay triangulation: let $\lambda_i^{(T)}$ be the i 'th eigenvalue of the finite element Laplacian. Then we have

$$\lambda_i^{(T)} \geq \lambda_i^{(T_D)} \quad \forall T, i$$

Computing intrinsic Delaunay triangulations

no lifting picture

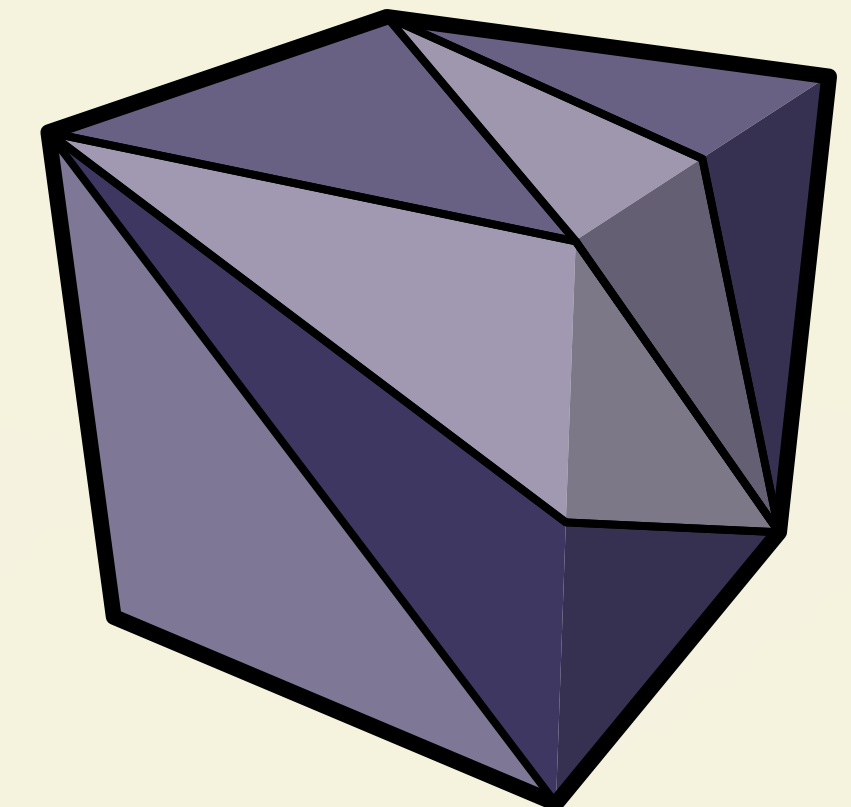
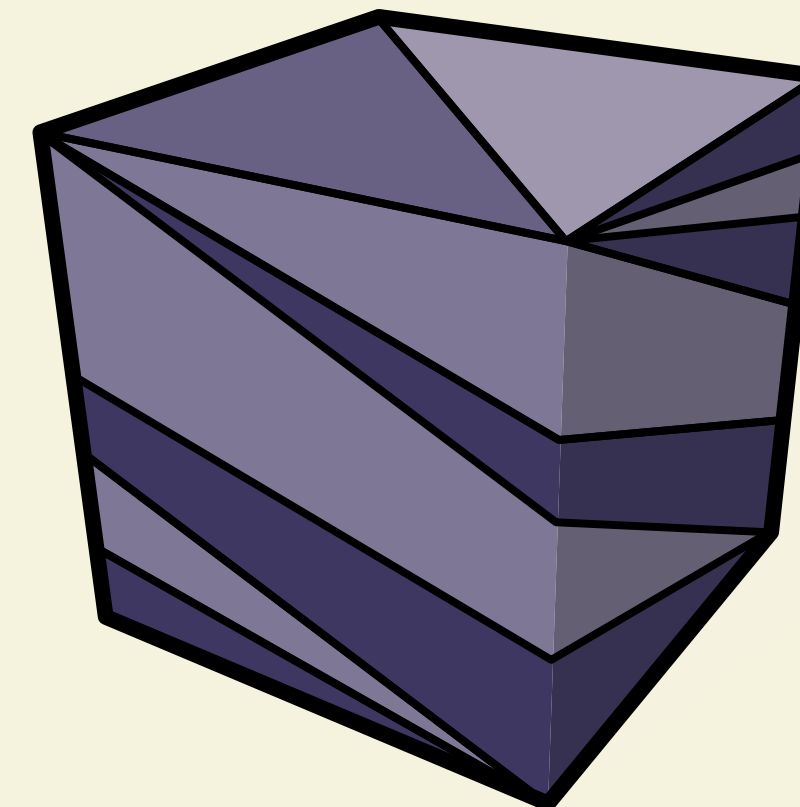
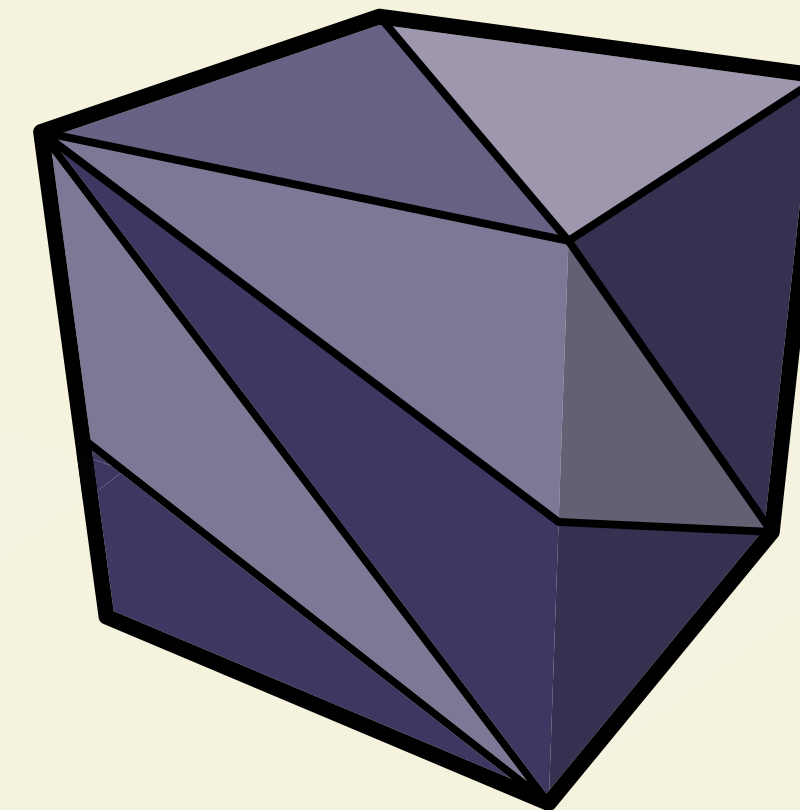
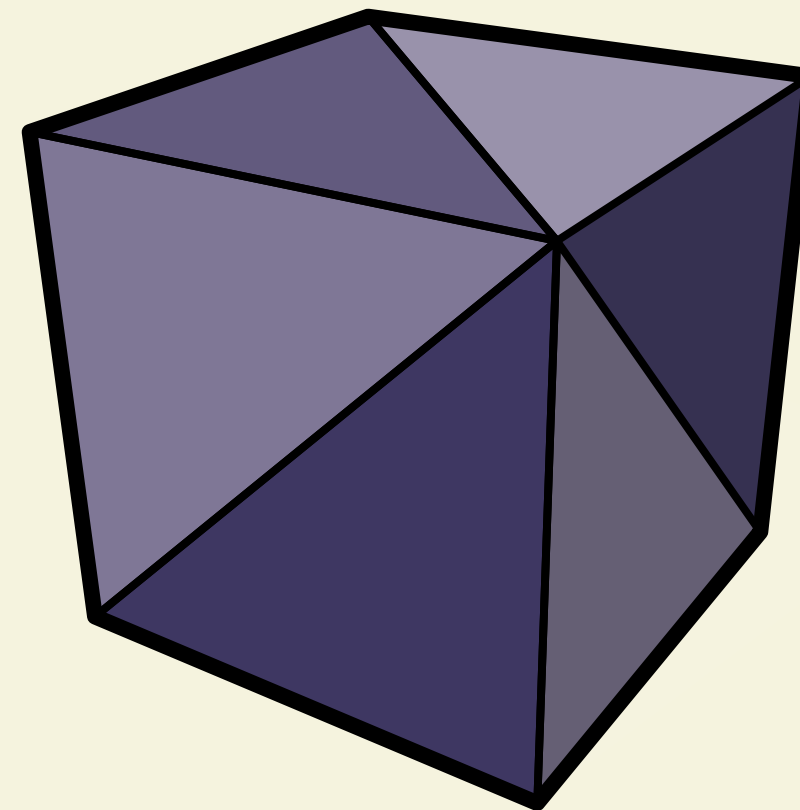
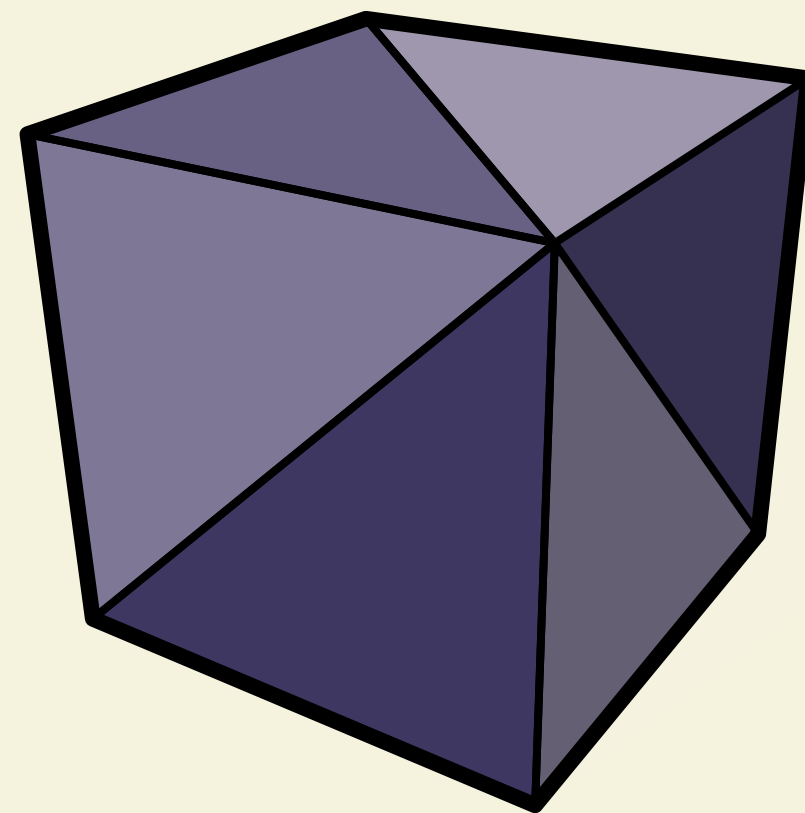
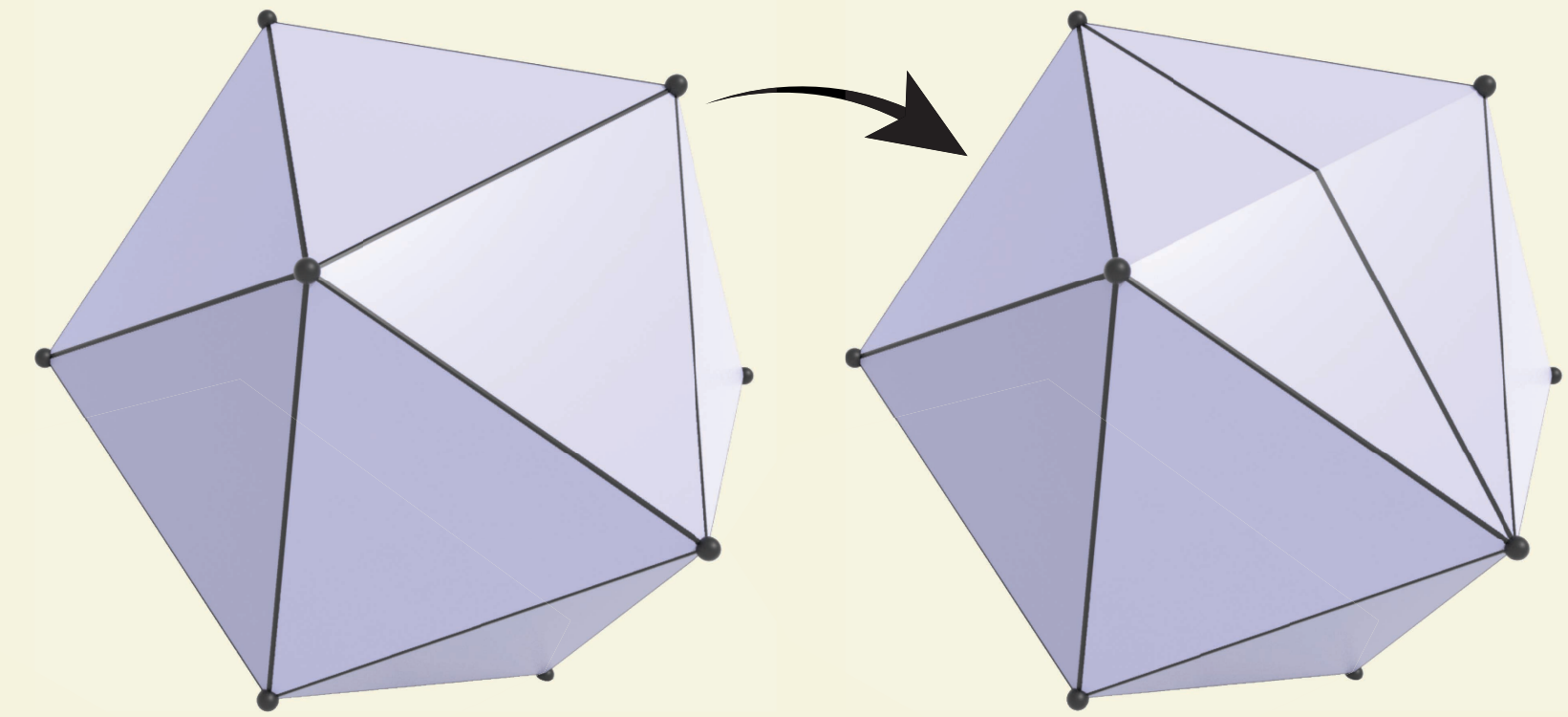


no $O(|V| \log |V|)$
convex hull algorithm

The intrinsic flip algorithm

[Bobenko & Springborn 2007]: runs in finite time

but it can take arbitrarily long!

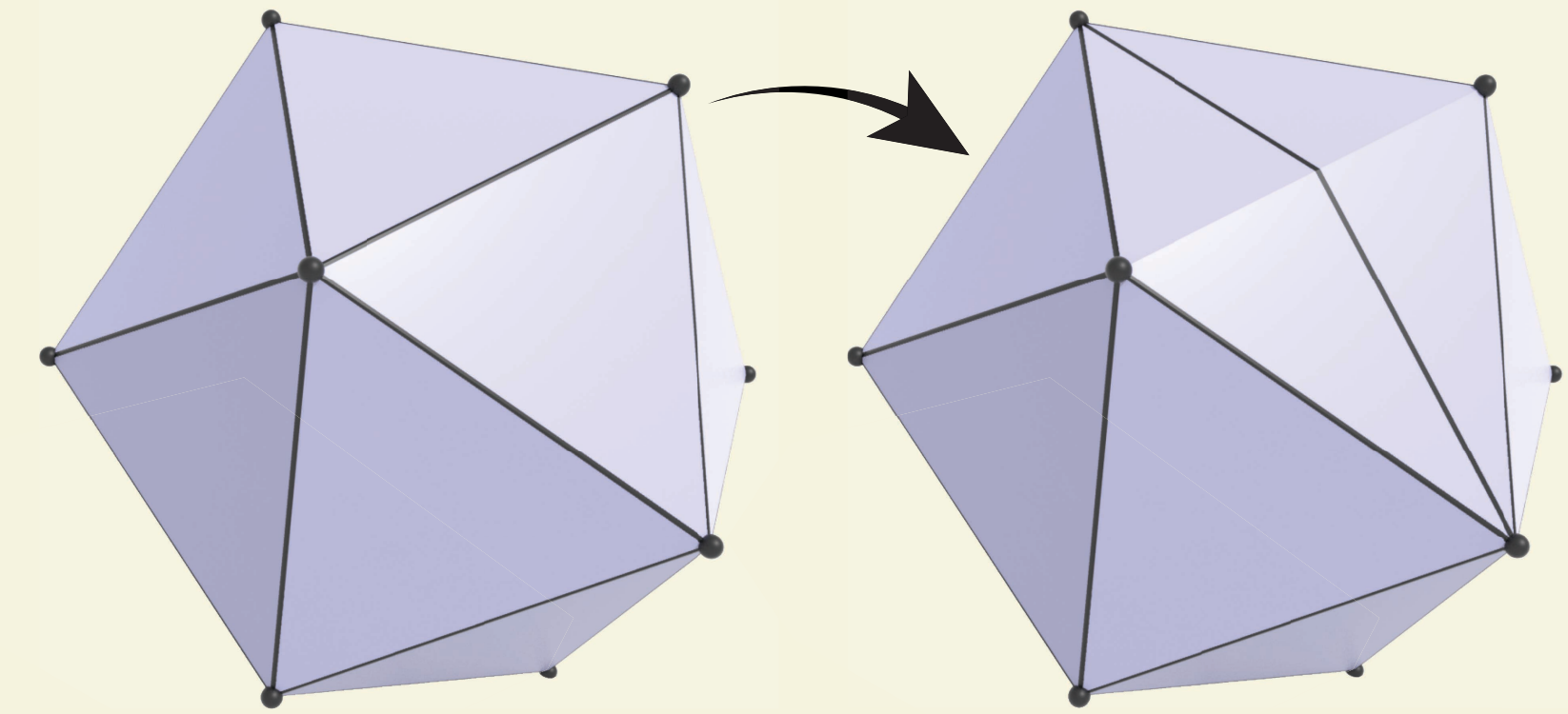


...

$|V|$ inadequate as a measurement
of problem difficulty

The intrinsic flip algorithm

termination



Theorem [Indermitte *et al.* 2001;
Bobenko & Springborn 2007]

The intrinsic flip algorithm
terminates after finitely
many flips.

Proof sketch:

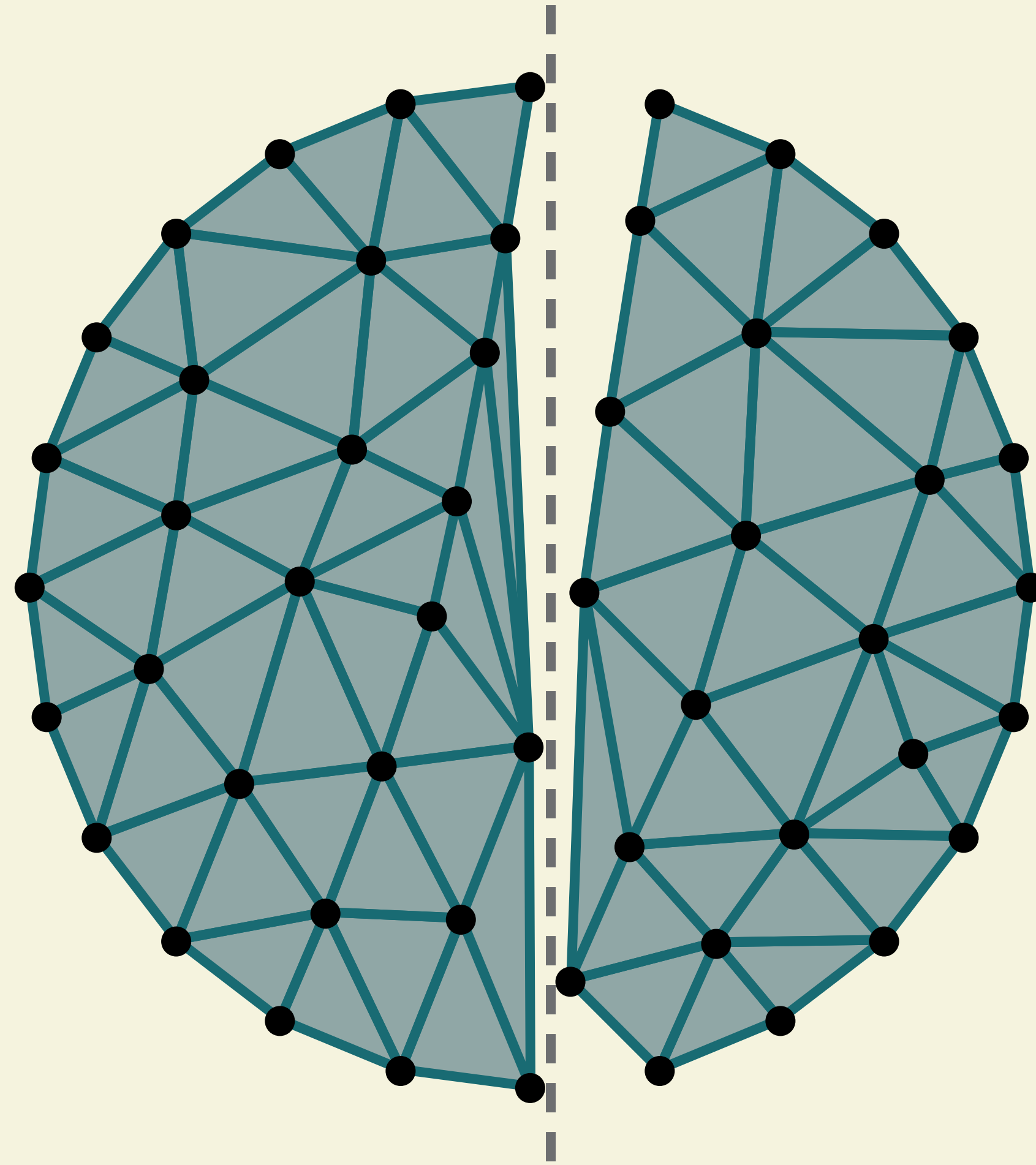
1. Find a function that decreases with each flip.
2. Prove that for any $r \in \mathbb{R}$, only finitely many triangulations have function value less than r .

Open Questions

faster algorithms

Can we use one of the faster algorithms?
(at least, in simple cases)

divide and conquer

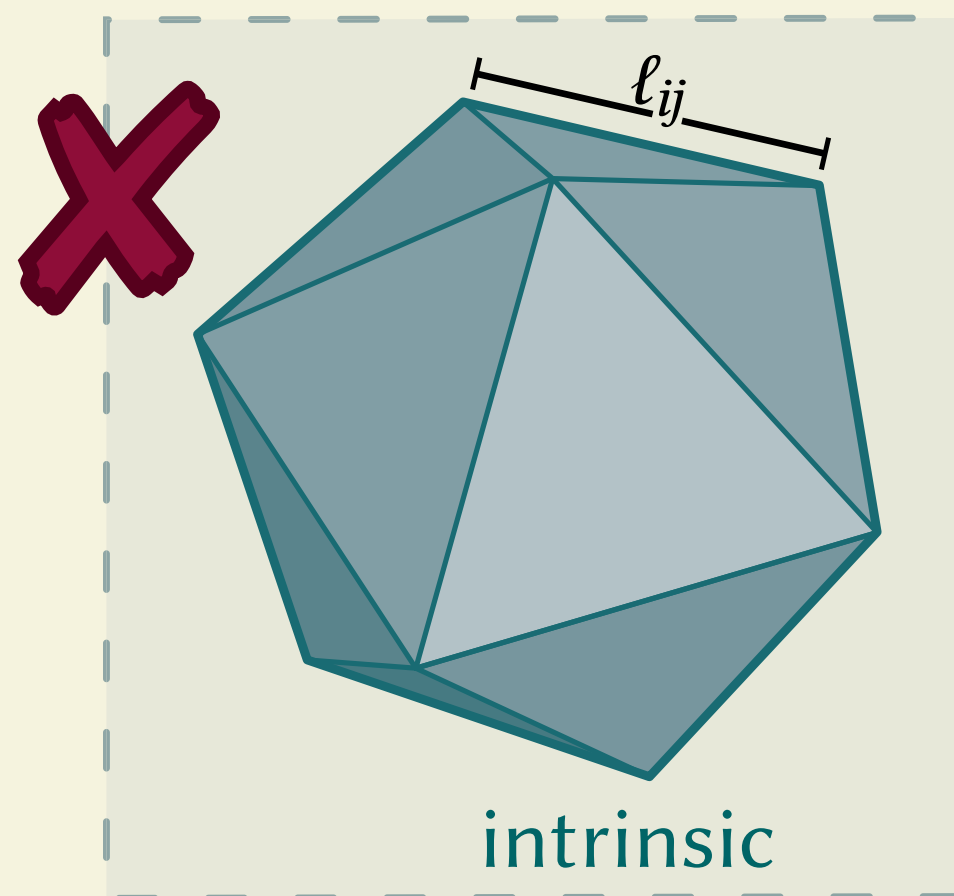


1. split in half
2. triangulate left
3. triangulate right

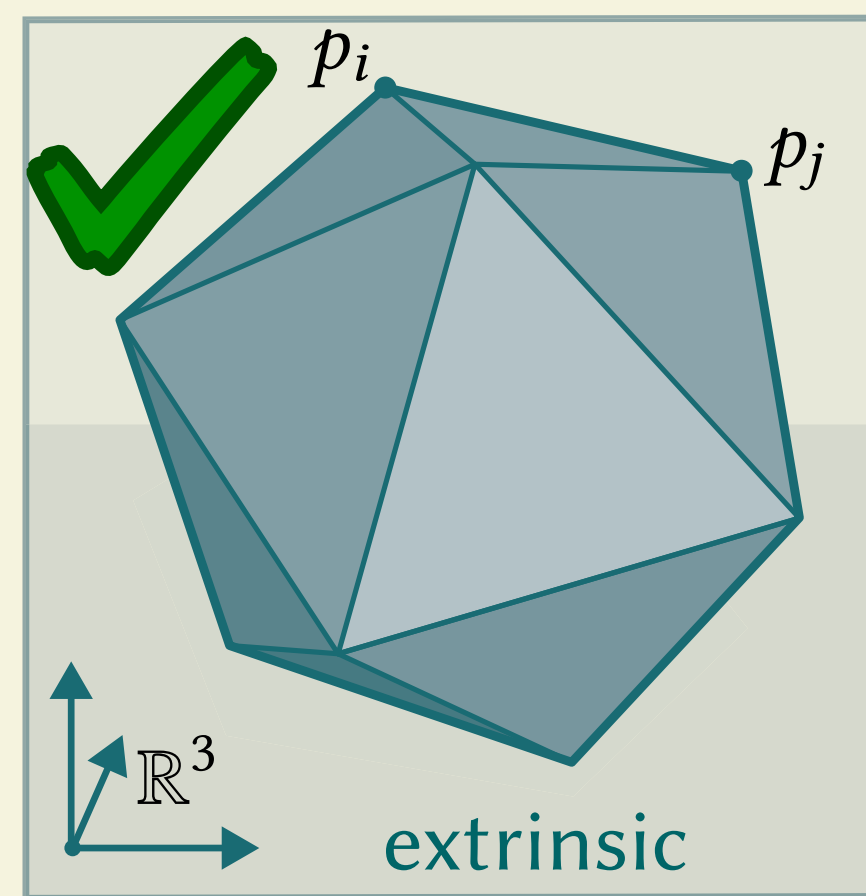
Open Questions

faster algorithms

Can we use one of the faster algorithms?
(at least, in simple cases)

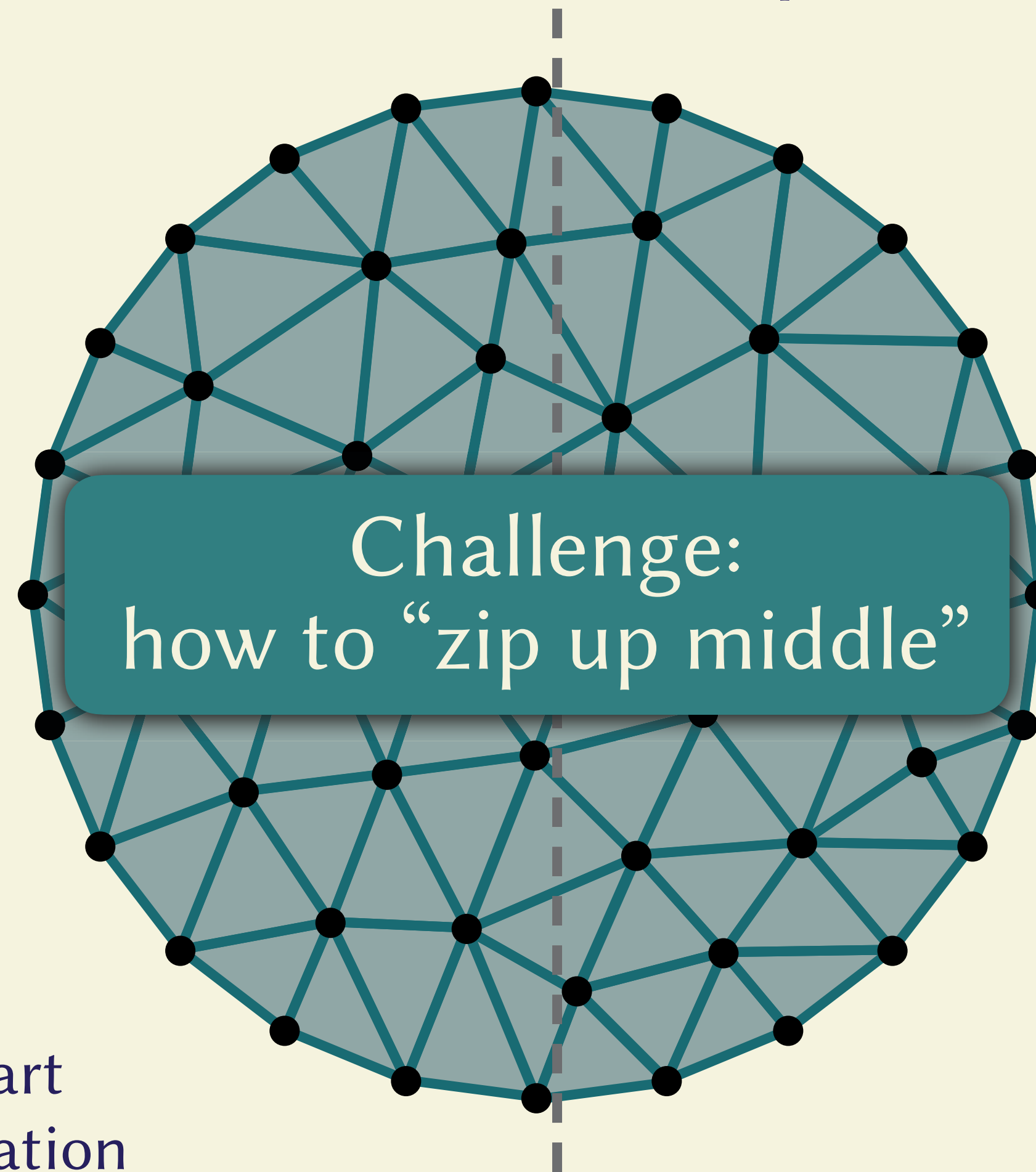


doesn't make sense for
an abstract intrinsic
triangulation



could work when we start
from an extrinsic triangulation
(common case)

divide and conquer



Challenge:
how to “zip up middle”

1. split in half
2. triangulate left
3. triangulate right
4. zip up middle

Open Questions

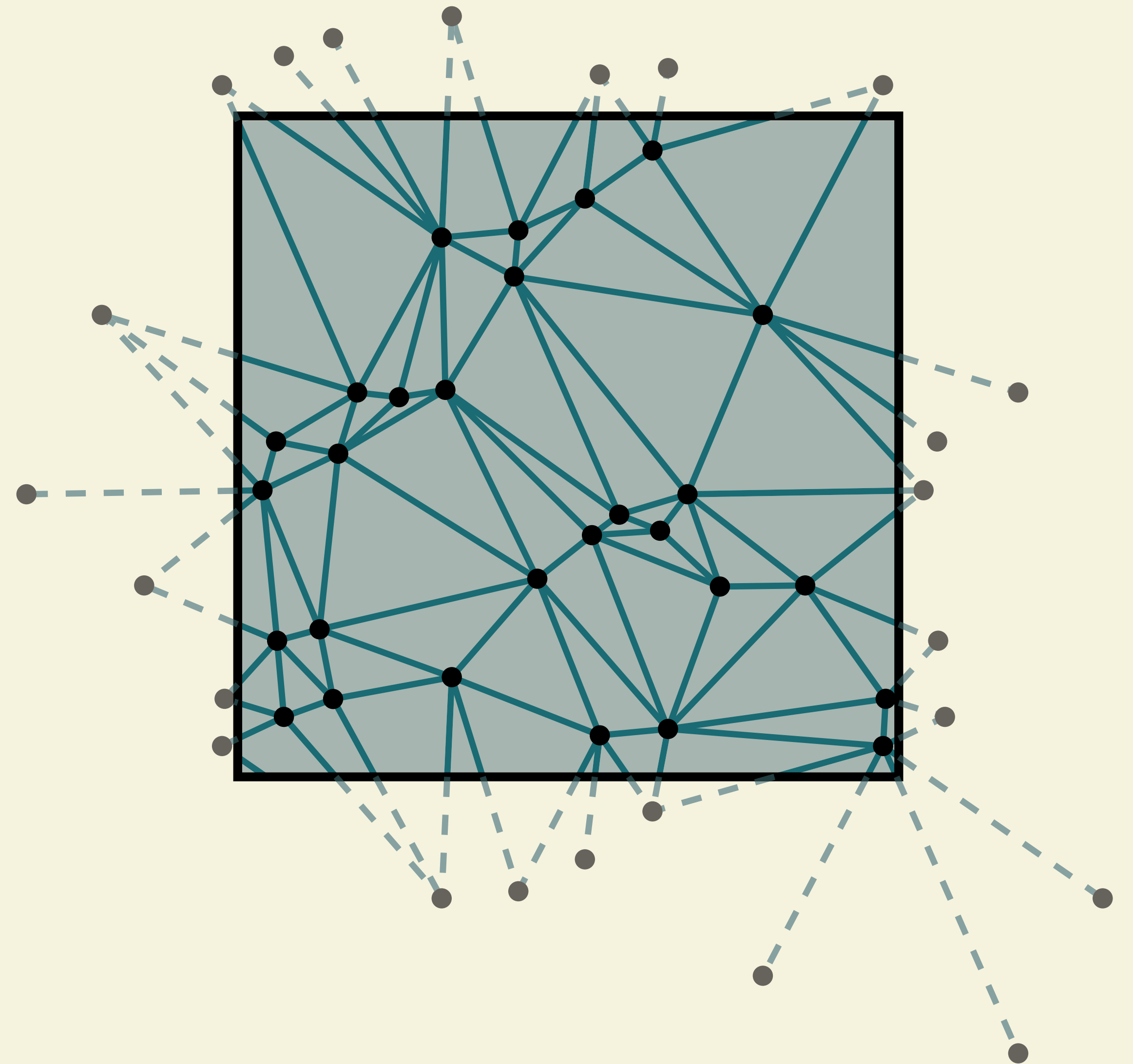
bounding flips

Can we bound the number of flips via another measurement of triangulation complexity?

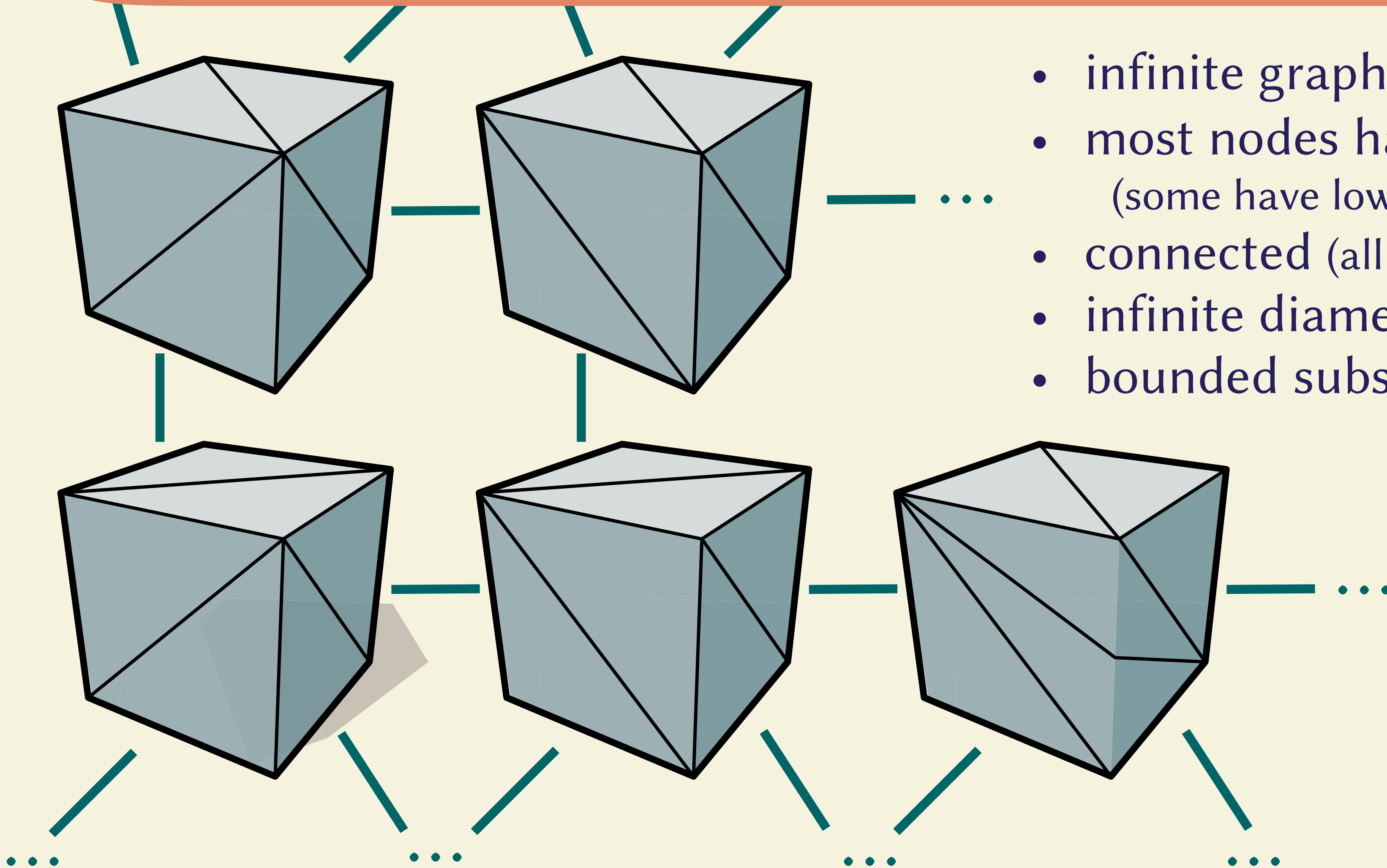
Theorem [Dubois 2023]

Fix a flat torus Γ . Starting from a triangulation T on Γ with vertex set $|V|$, the flip algorithm reaches the Delaunay triangulation within $C_\Gamma |V|^2 \ell_T$ flips, where ℓ_T is the length of the longest edge of T .

Can we bound the number of flips from an extrinsic triangulation?

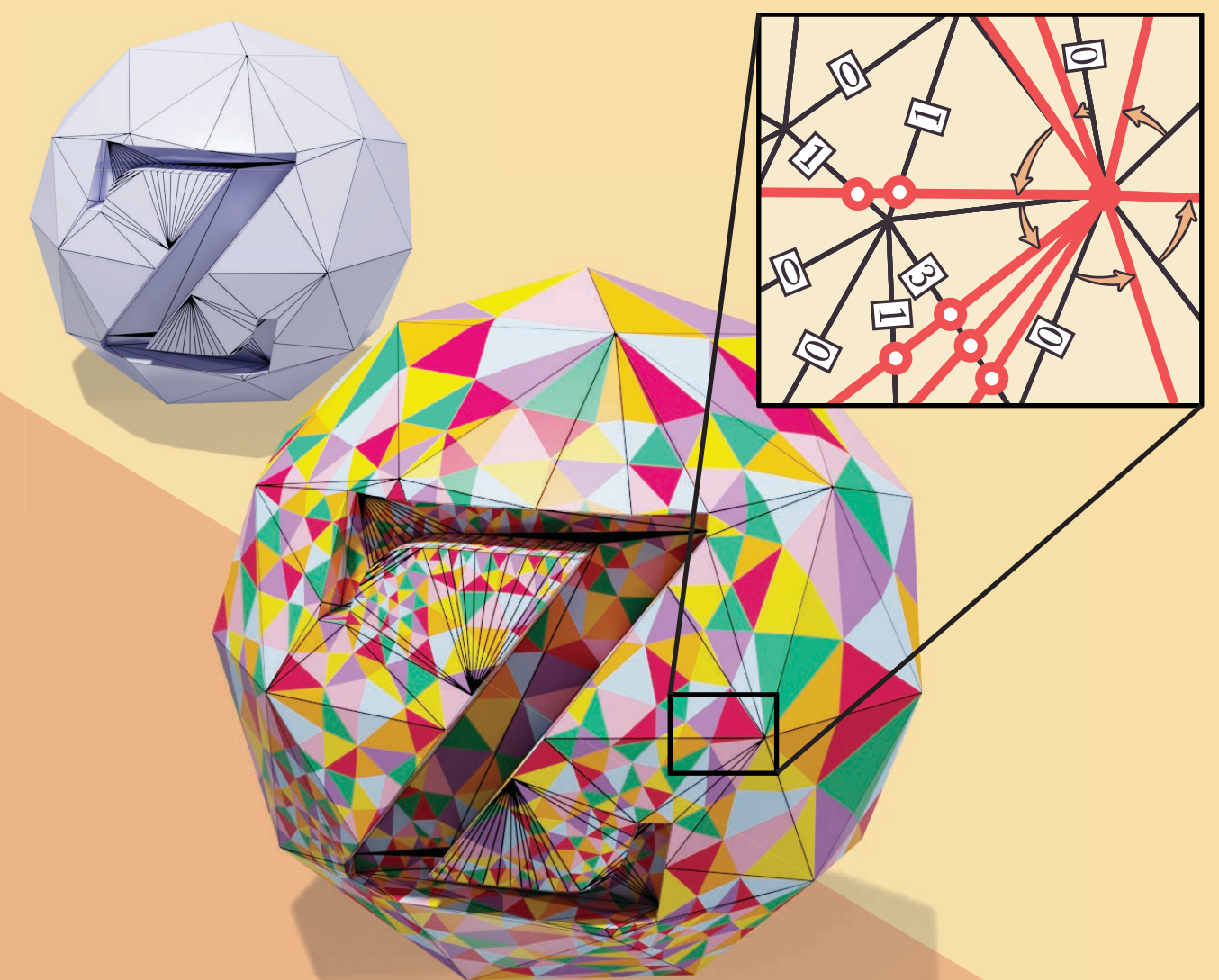


Alternative Perspective: the Flip Graph



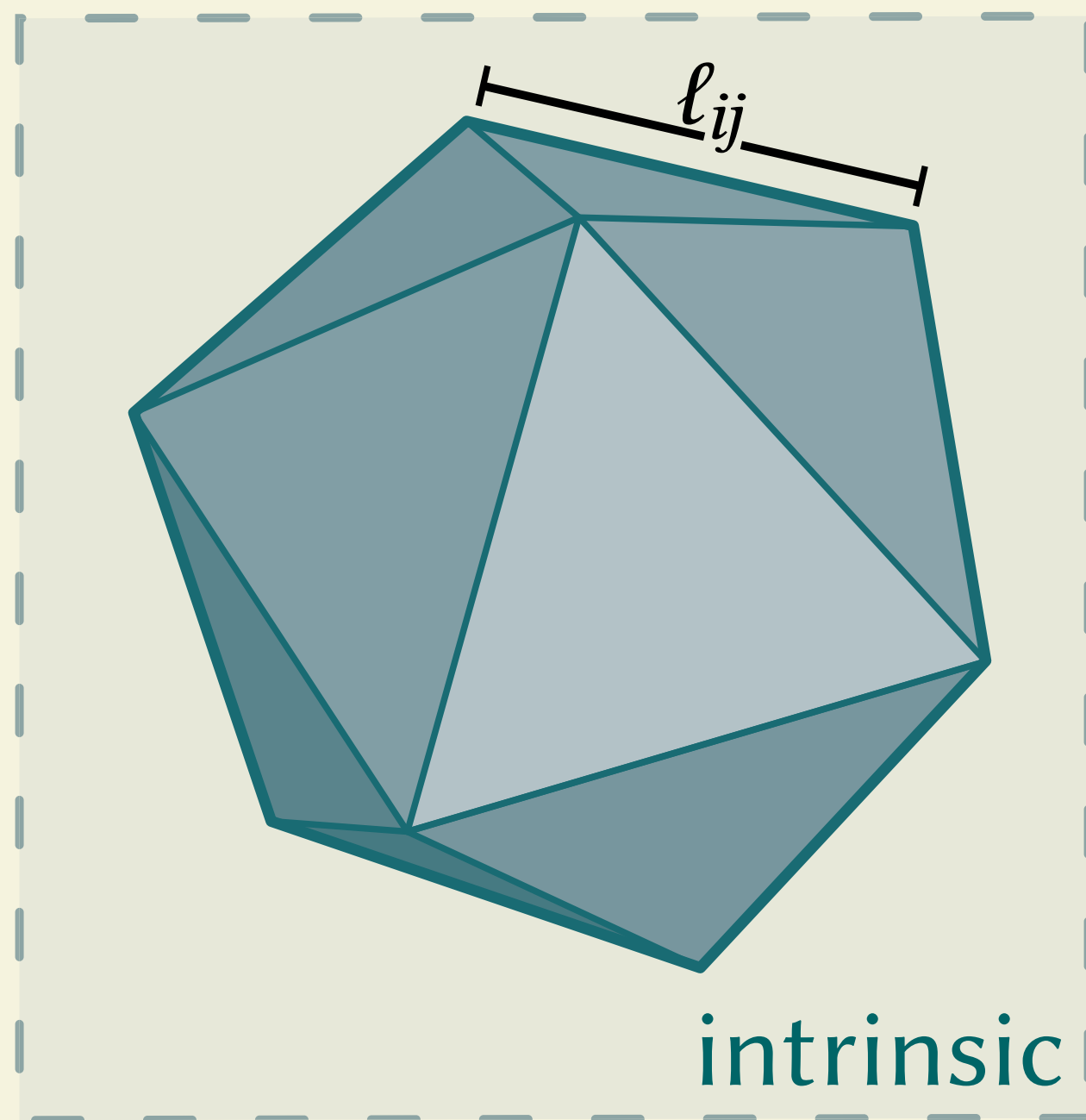
- infinite graph
- most nodes have degree $|E|$
(some have lower degree due to non-flippable edges)
- connected (all nodes connected to Delaunay)
- infinite diameter
- bounded subsets? (efficiently-flippable meshes)

III. Data Structures for Intrinsic Triangulations



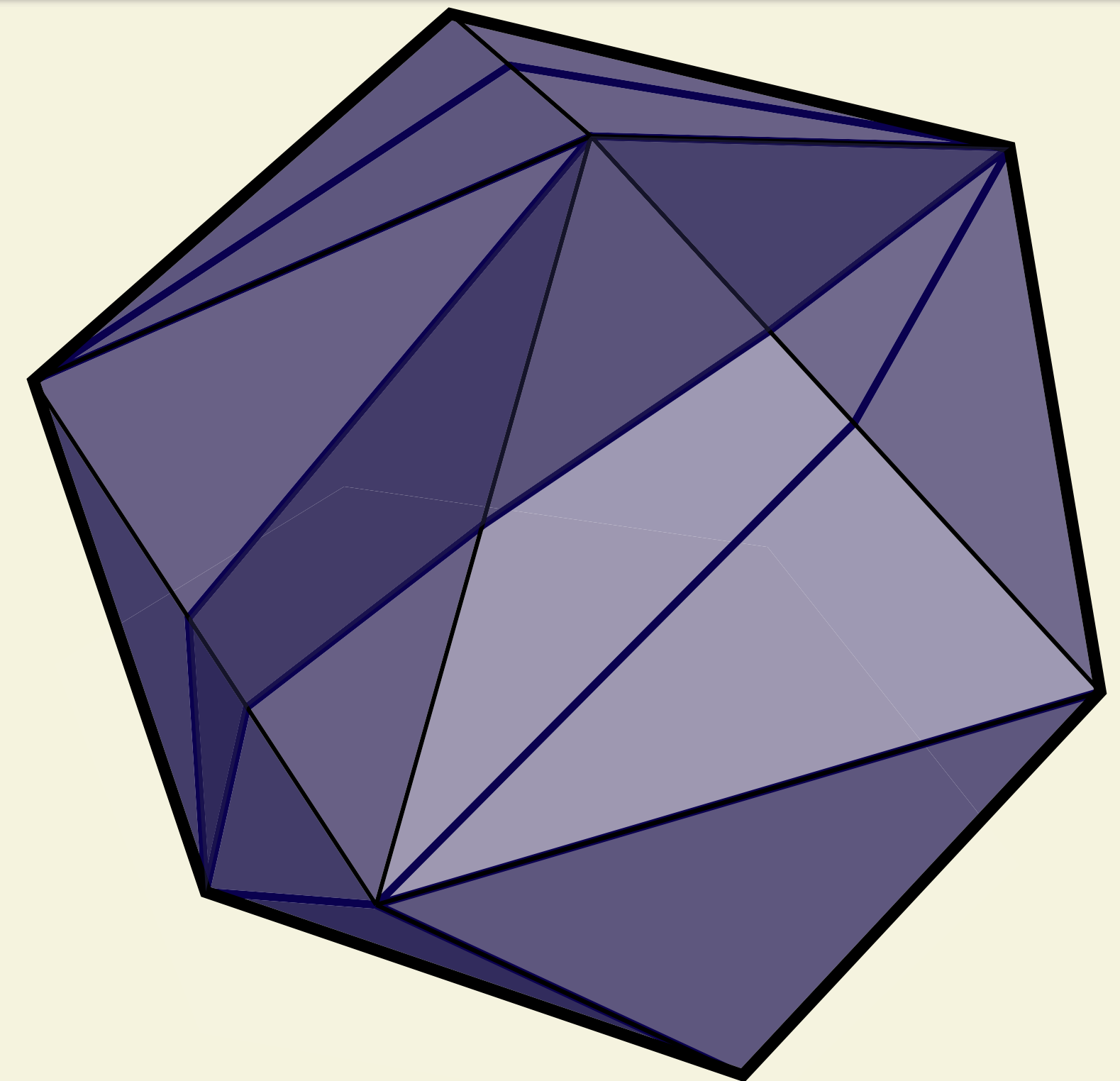
【 G., Sharp, & Crane. 2021. Integer Coordinates for Intrinsic Geometry Processing. *ACM Transactions on Graphics* 】

The abstract intrinsic triangulation



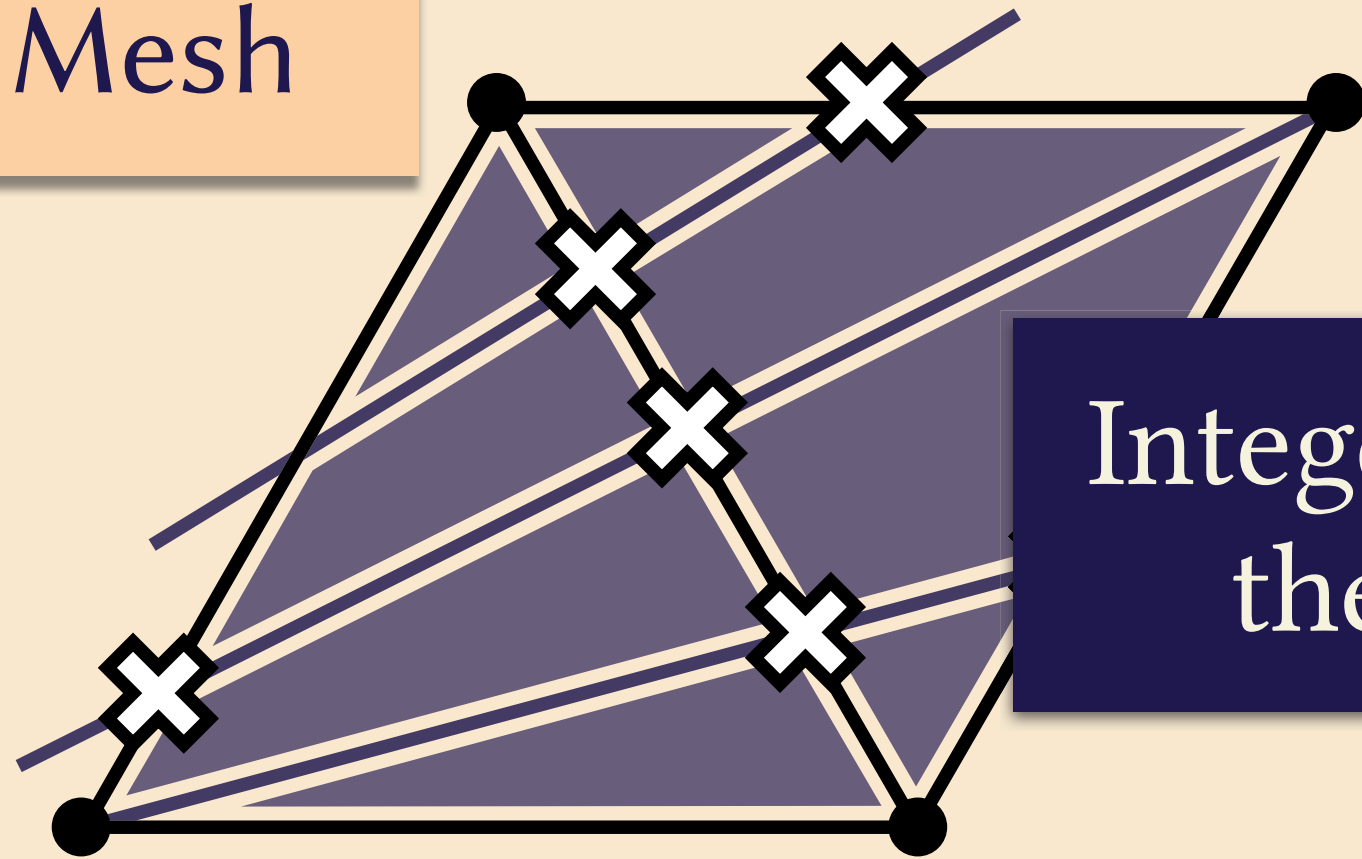
vertex set V
edge set E
face set F
edge lengths $\ell : E \rightarrow \mathbb{R}$

Challenge: correspondence
with an extrinsic triangulation



Correspondence data structures

Overlay Mesh

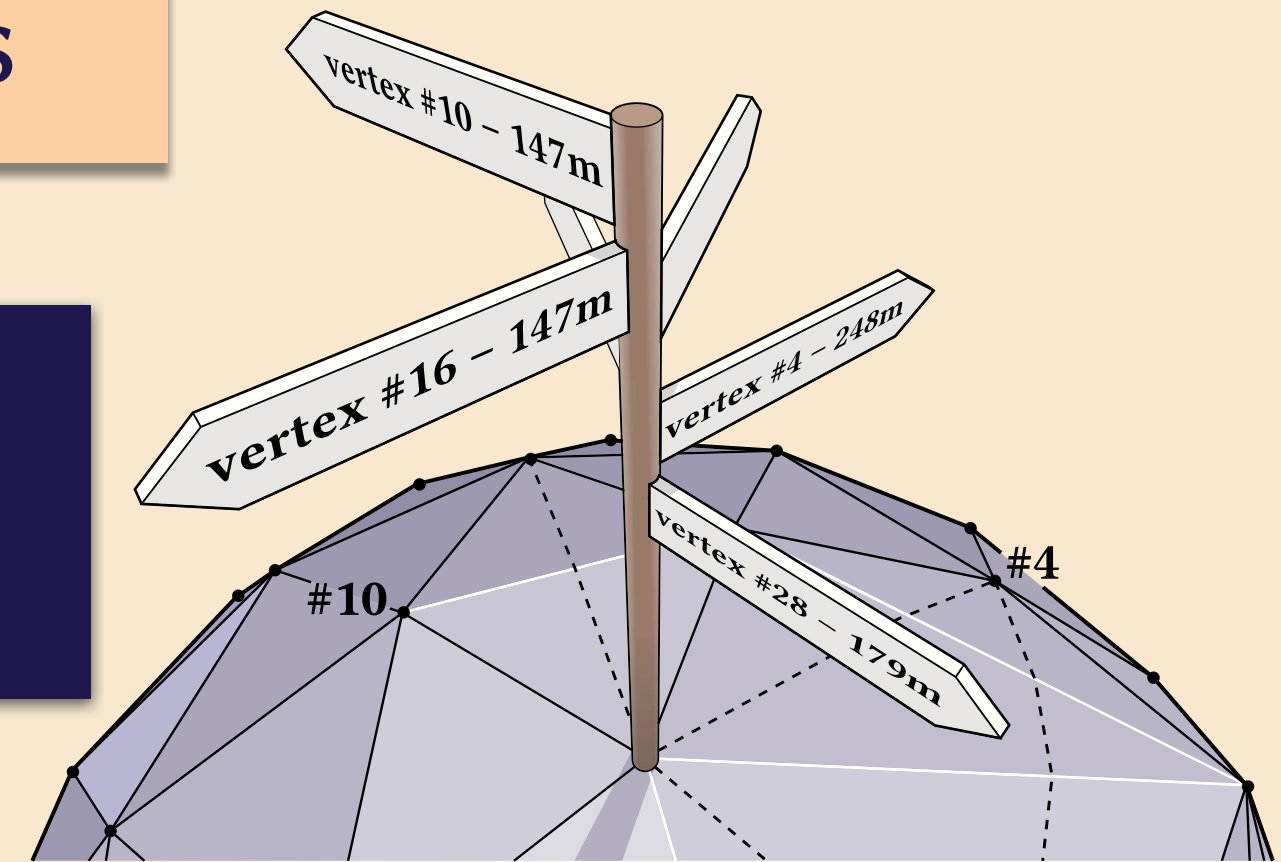


Integer coordinates combine the best of both worlds

[Fisher, Springborn, Bobenko & Schröder 2006]

- Explicit mesh of common subdivision
- Edge flips nonlocal & expensive
- **No further operations**

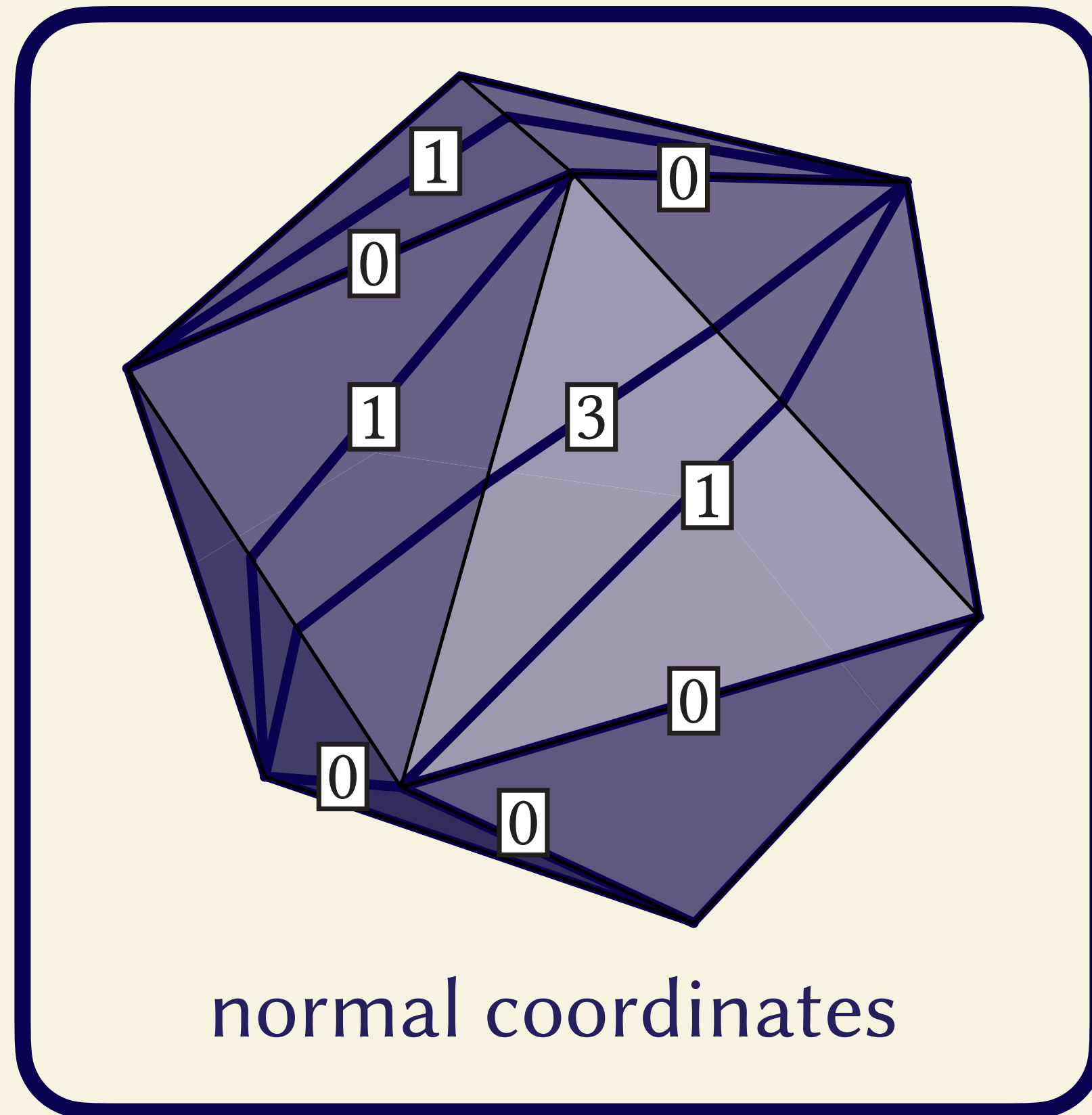
Signposts



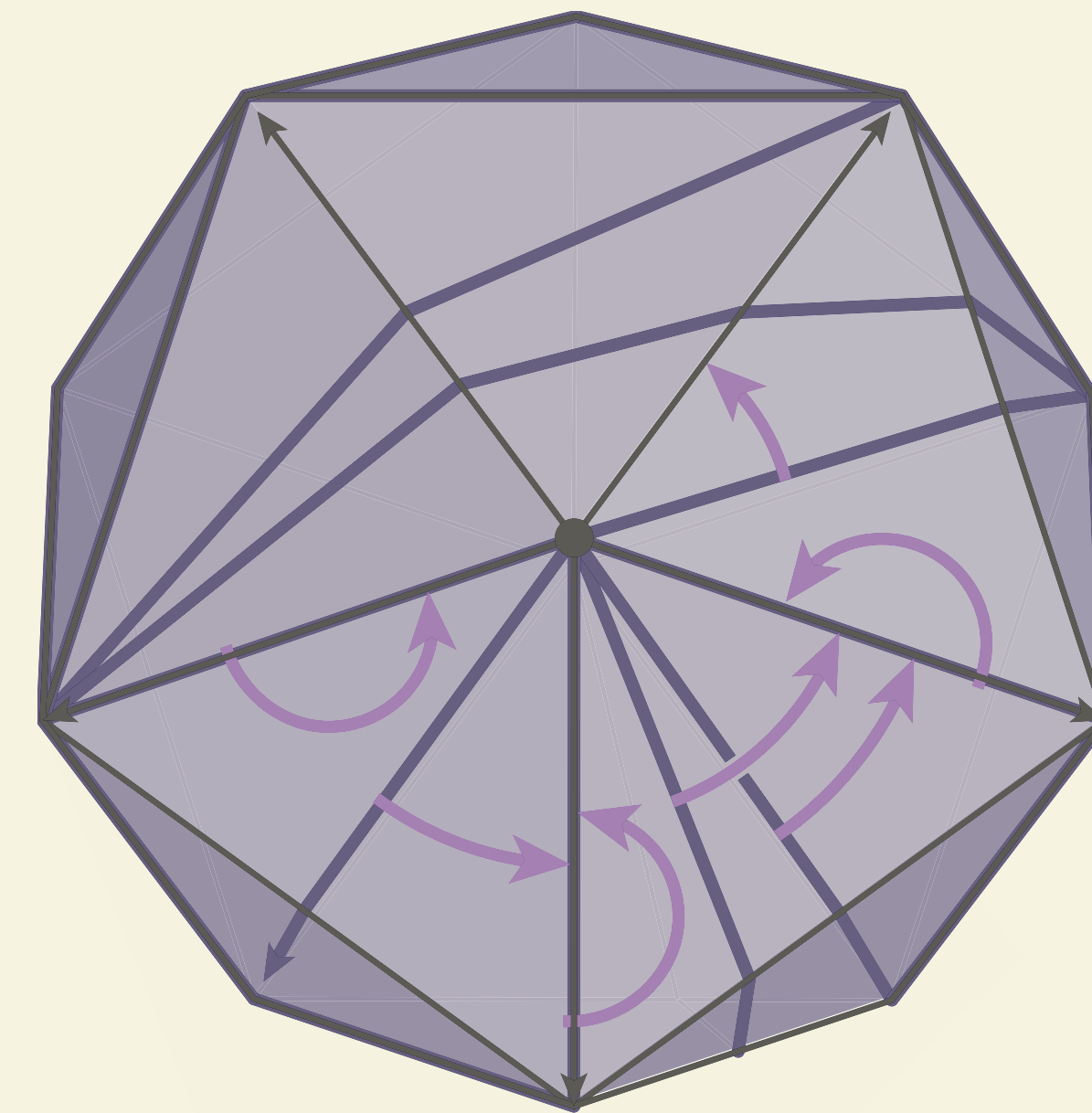
[Sharp, Soliman & Crane 2019]

- Floating point *signpost* vectors at vertices
- **Supports many local mesh operations**
- Common subdivision connectivity may be invalid

The integer coordinates data structure

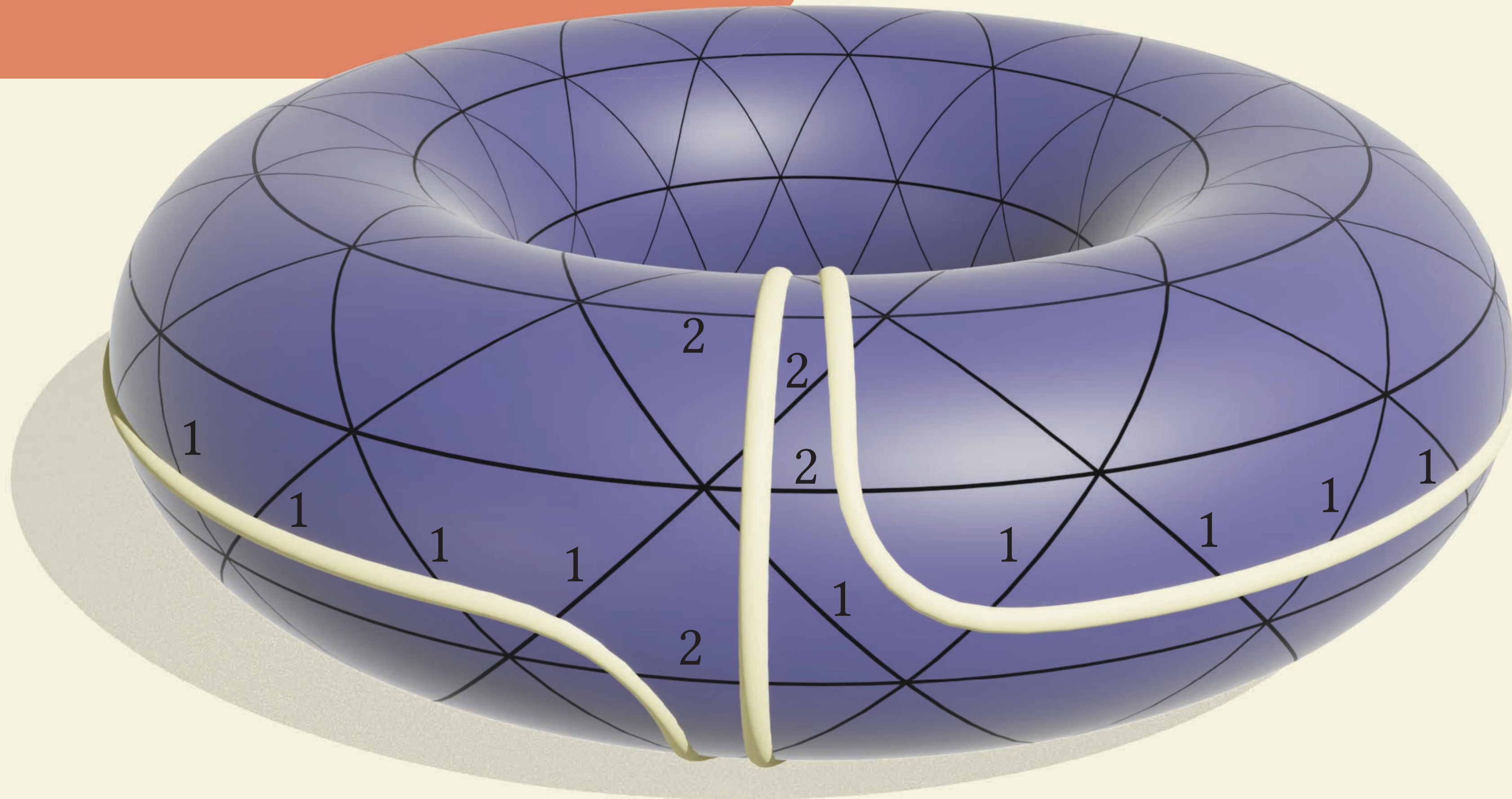


+



(concretely, just 3 integers per mesh edge)

Normal coordinates



Foundations: [Kneser 1929; Haken 1961]

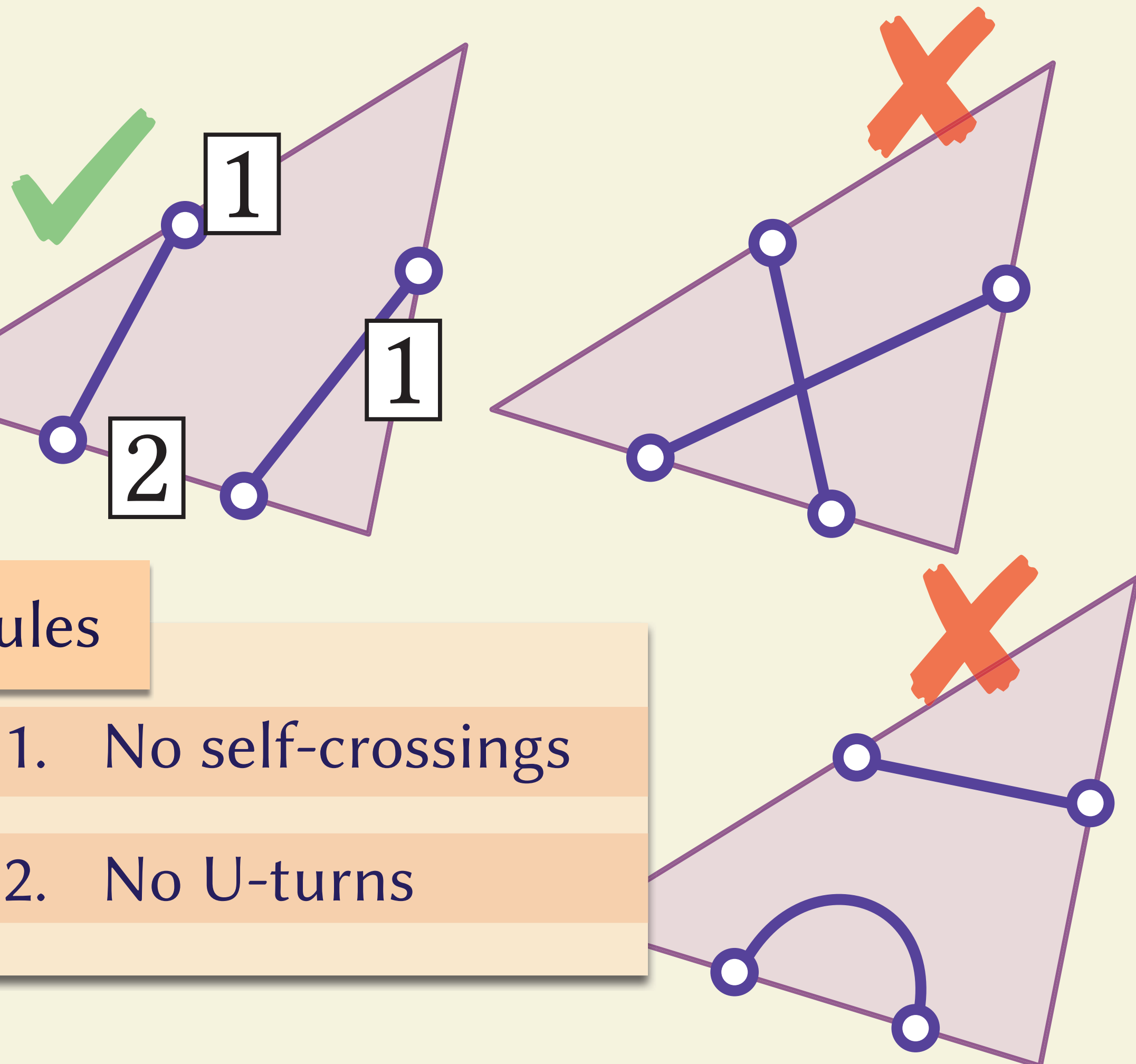
Computational Topology: [Schaefer+ 2008; Erickson & Nayyeri 2013]

Geometry Processing: [Hass & Trnkova 2020]

Encoding a curve with normal coordinates

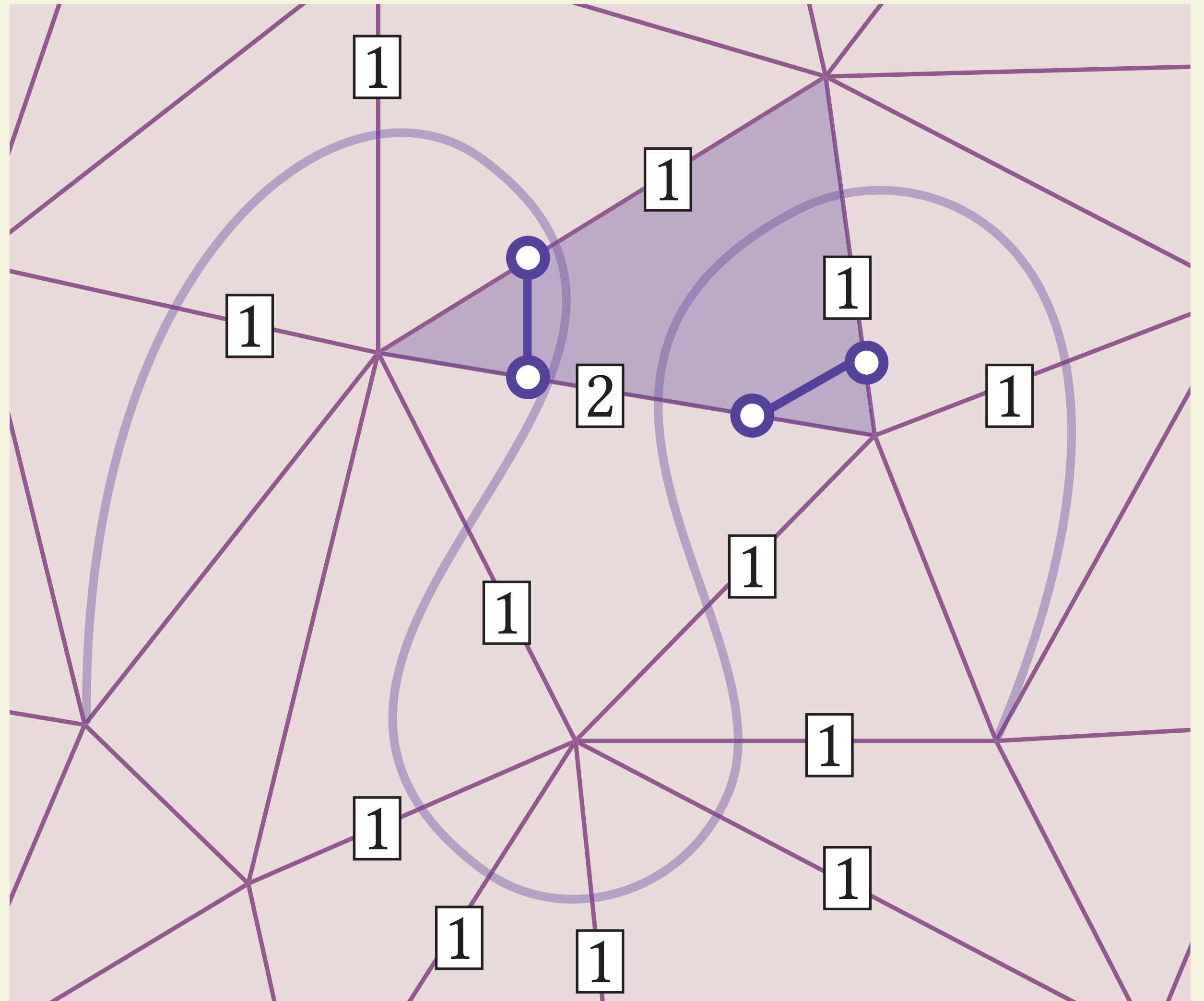
- Represents curve up to homotopy (on the surface punctured at vertices)
- Equivalently, encodes a sequence of triangles

Encoding a curve with normal coordinates



Rules

1. No self-crossings
2. No U-turns

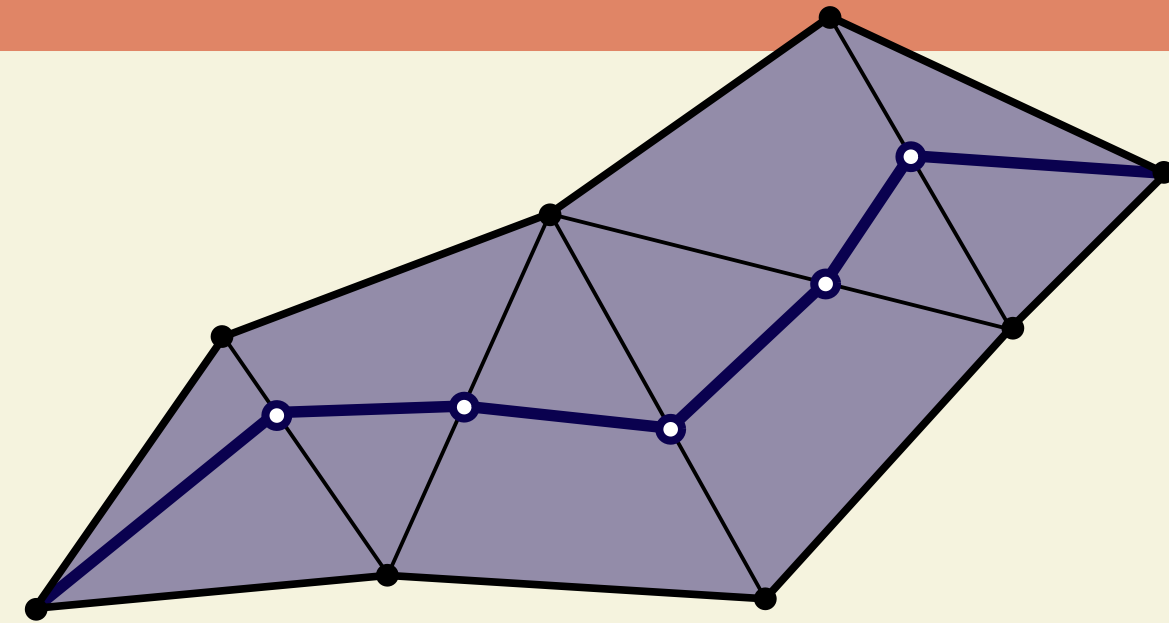


Encoding a curve with normal coordinates

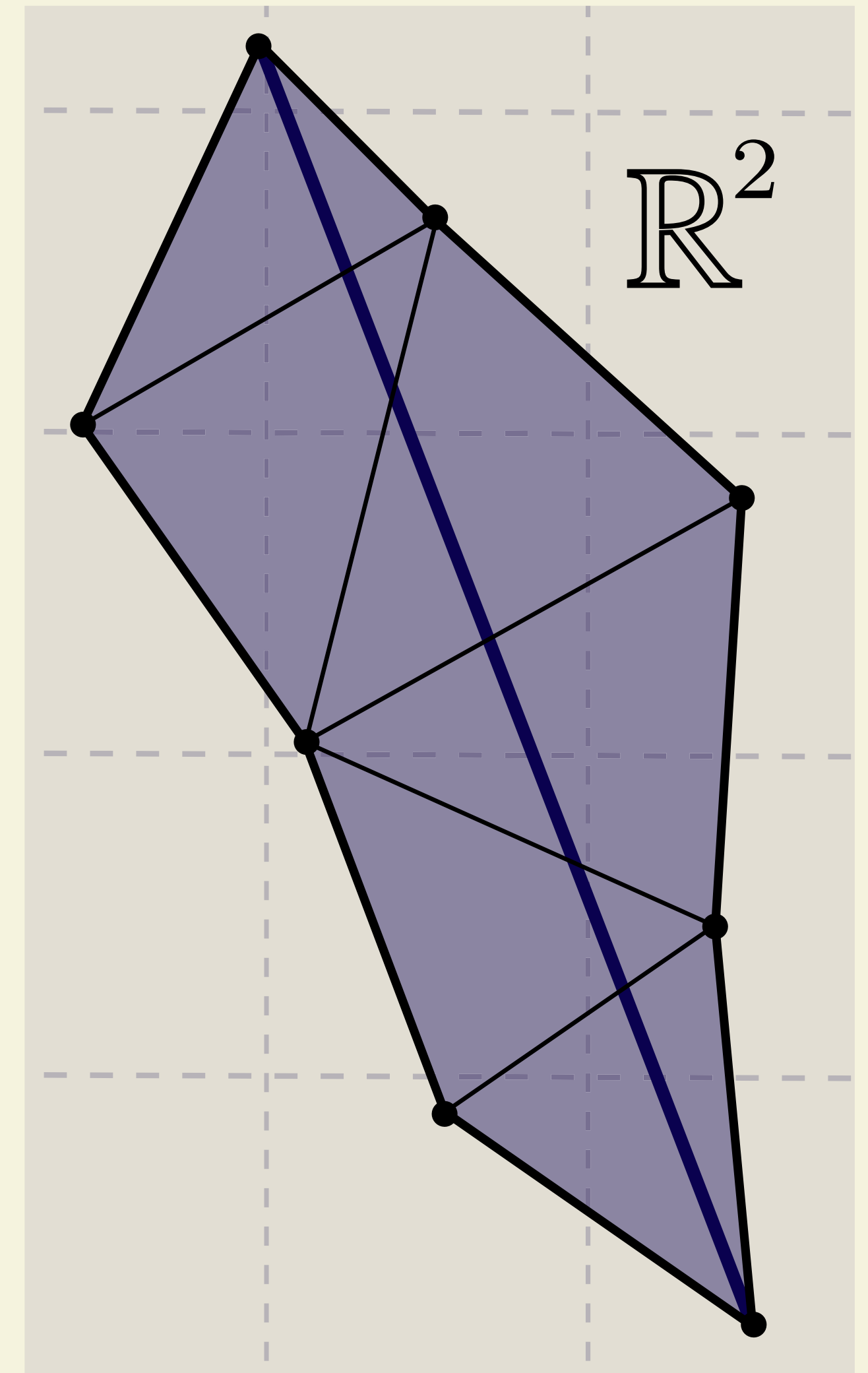
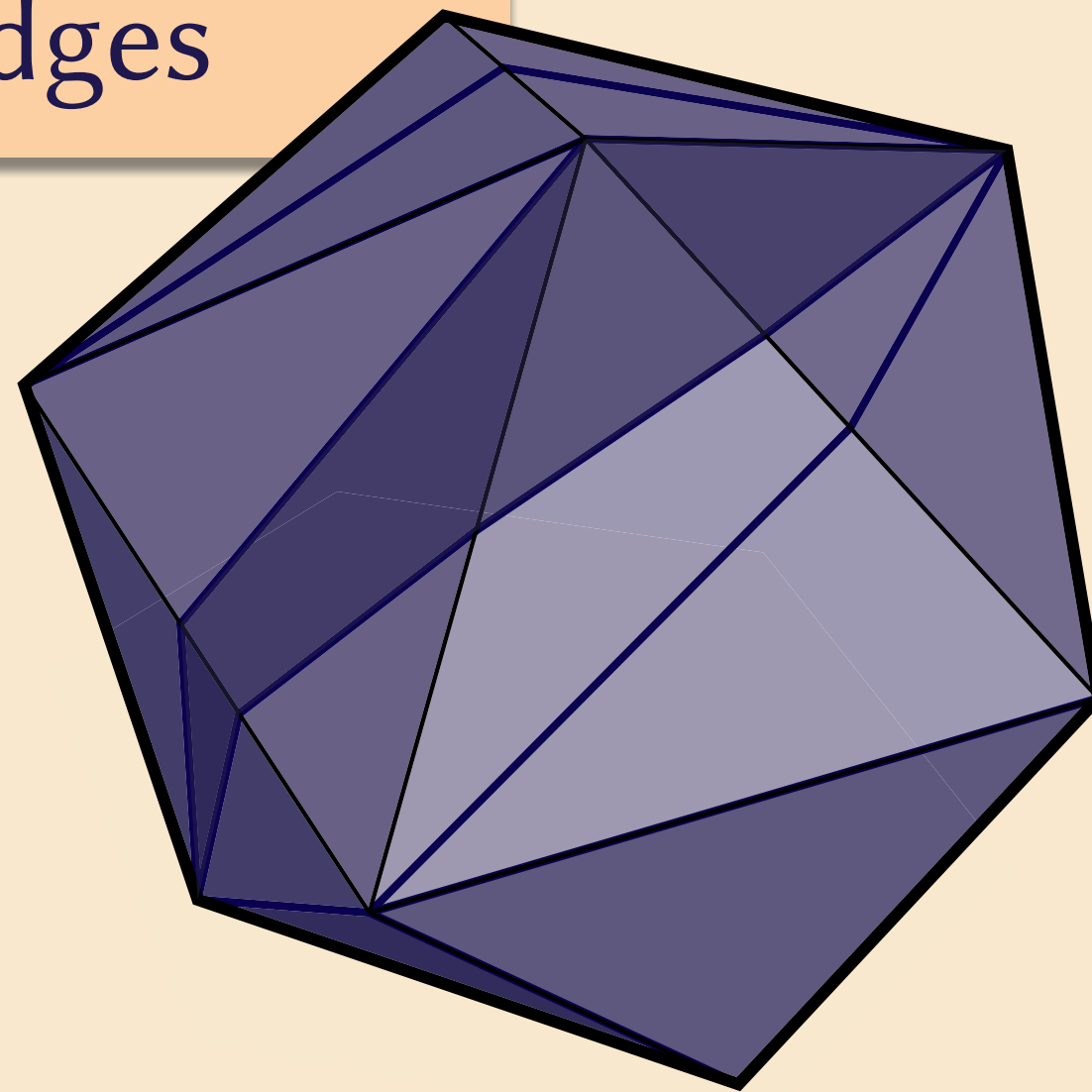
- Normality conditions determine curves within each triangle

Finding the exact curve geometry

- So far: triangle strip
- True curve is *geodesic*
 - Lay out in plane to find exact curve
- Normal coordinates determine edges exactly



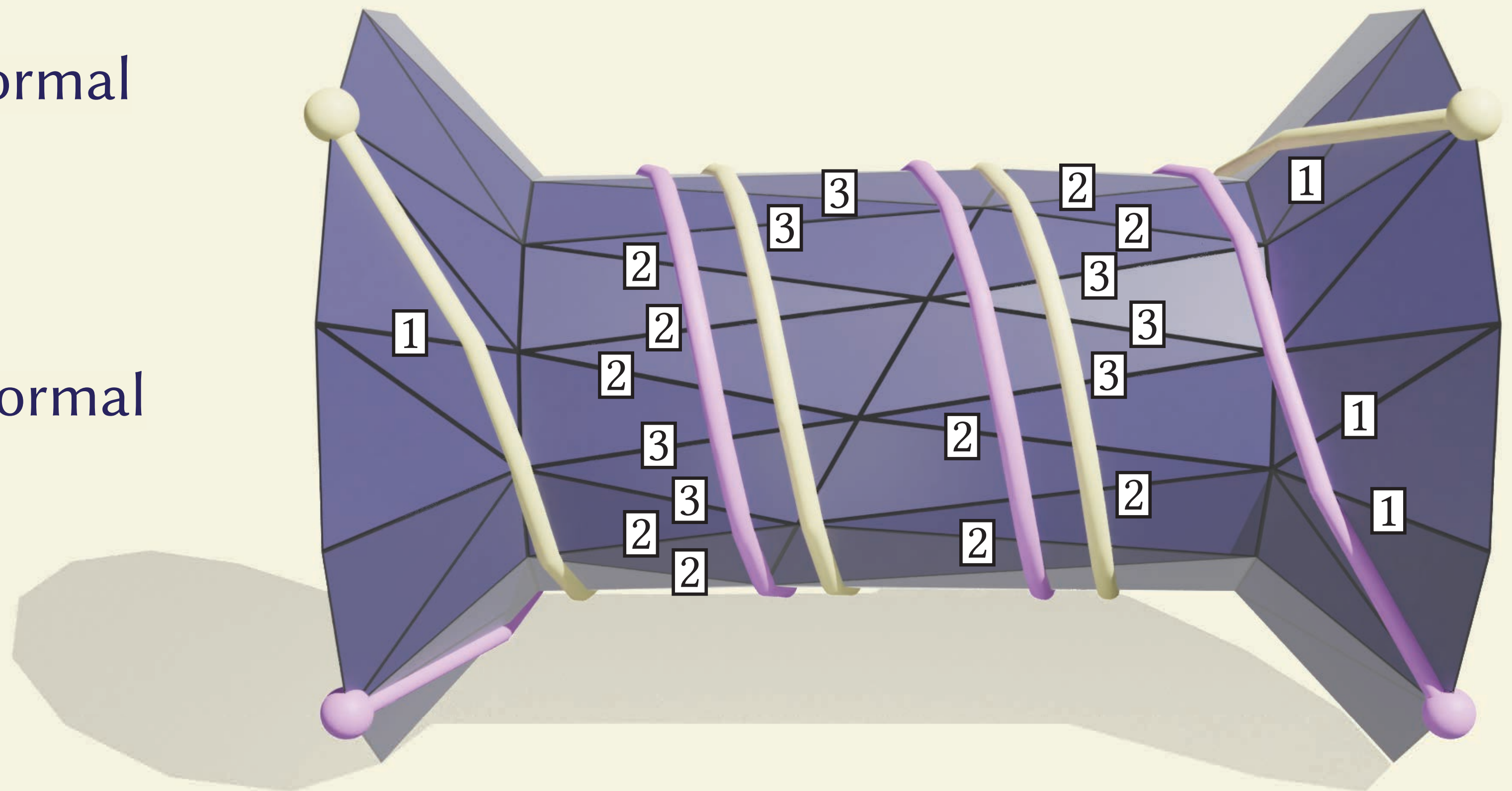
Intrinsic
edges



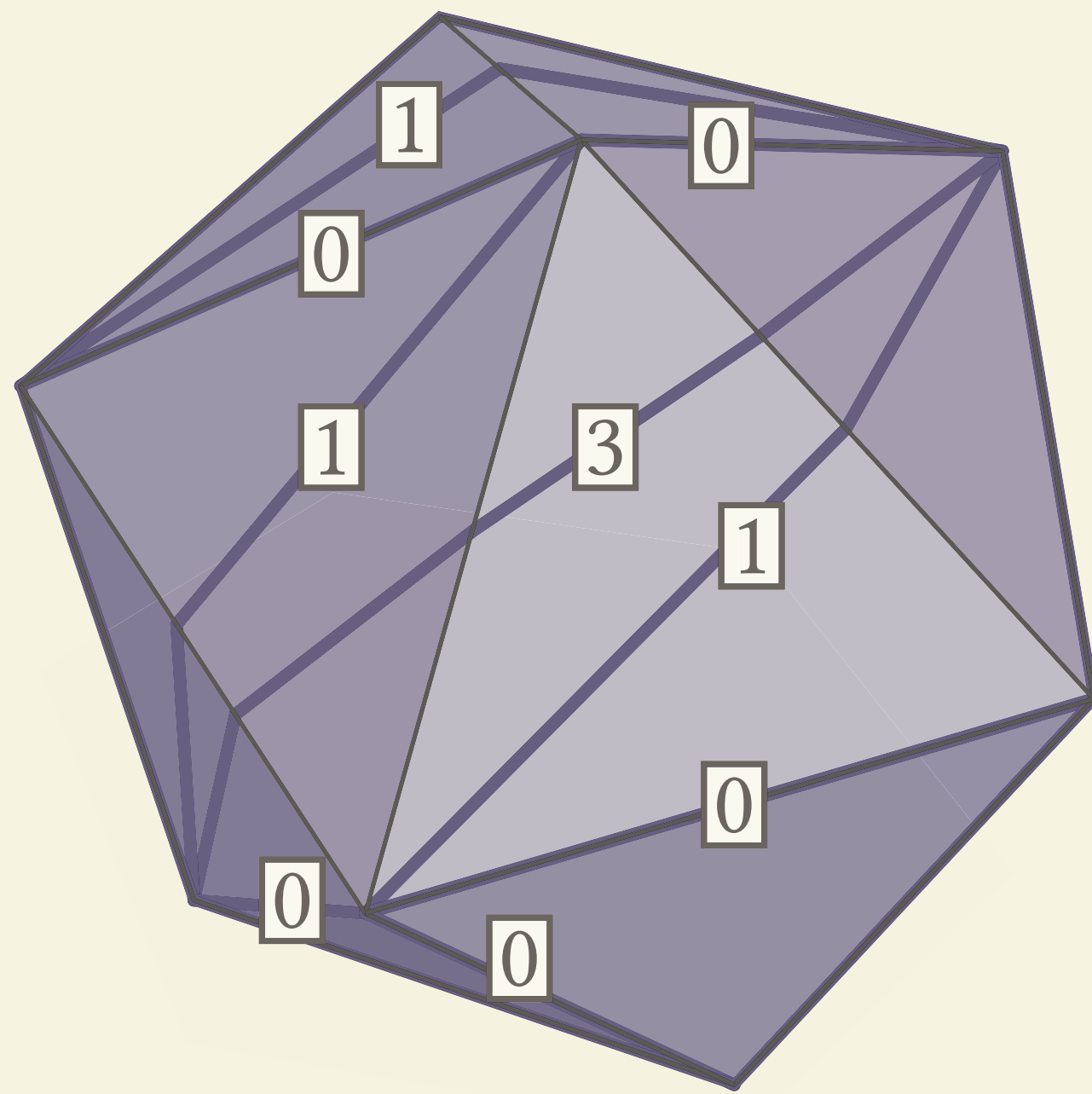
Collections of Curves

- e.g. edges of a triangulation
- Could store multiple sets of normal coordinates
 - ▶ Expensive 
- Instead, just store one set of normal coordinates

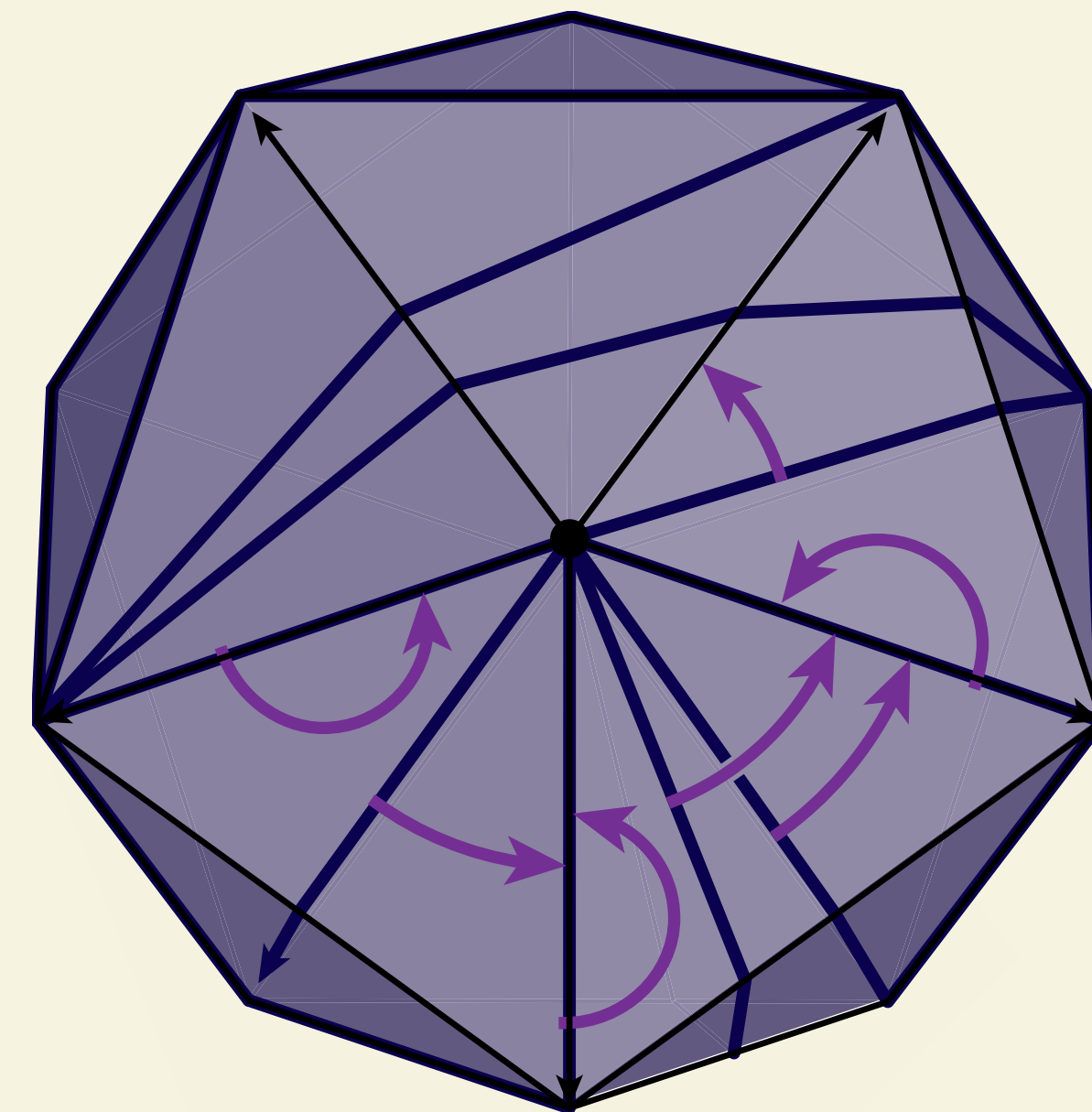
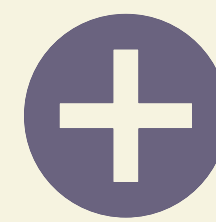
Store entire triangulation
using one integer per edge



The integer coordinates data structure



normal coordinates

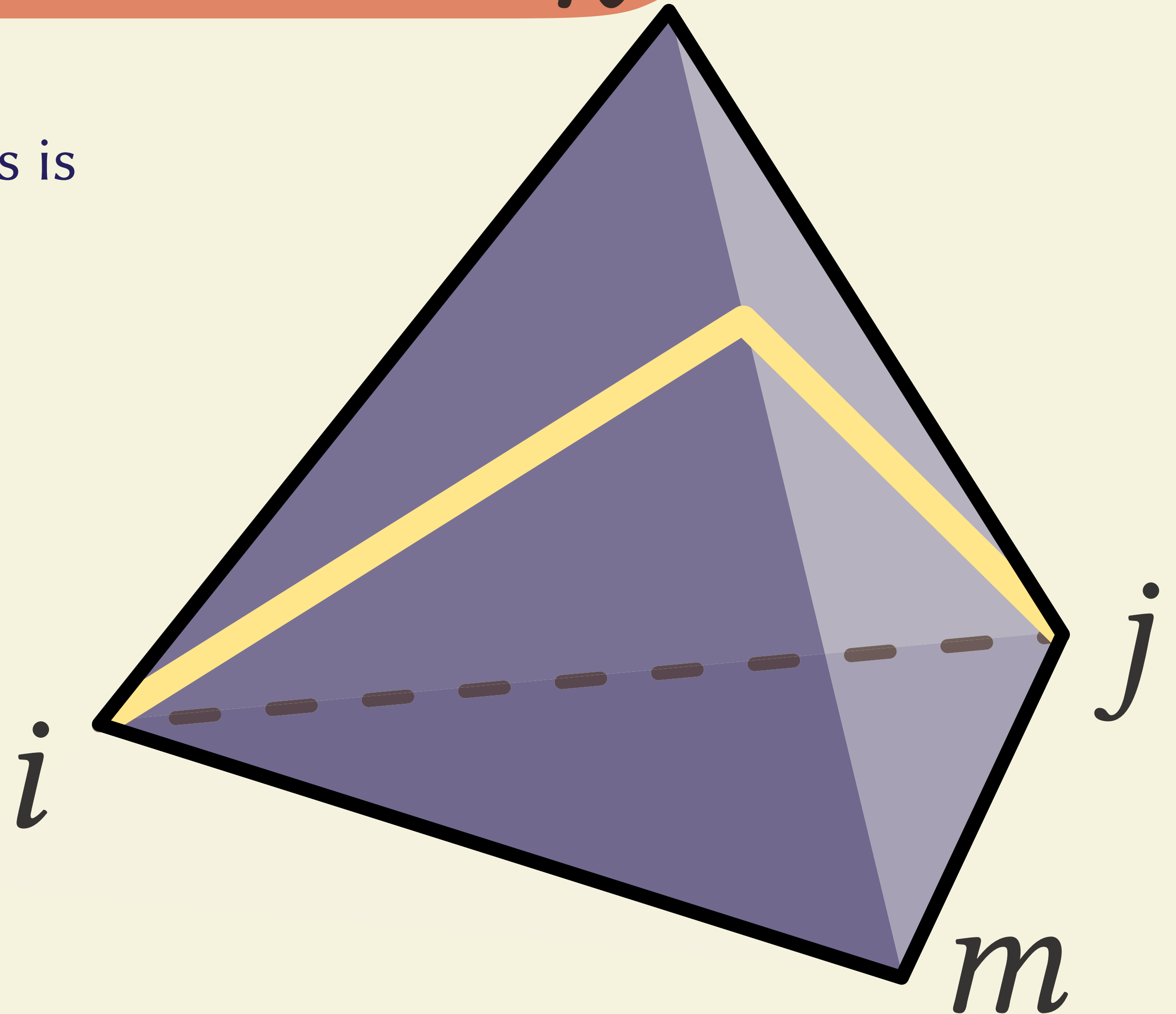


roundabouts

Normal coordinates are not enough to encode correspondence

 k

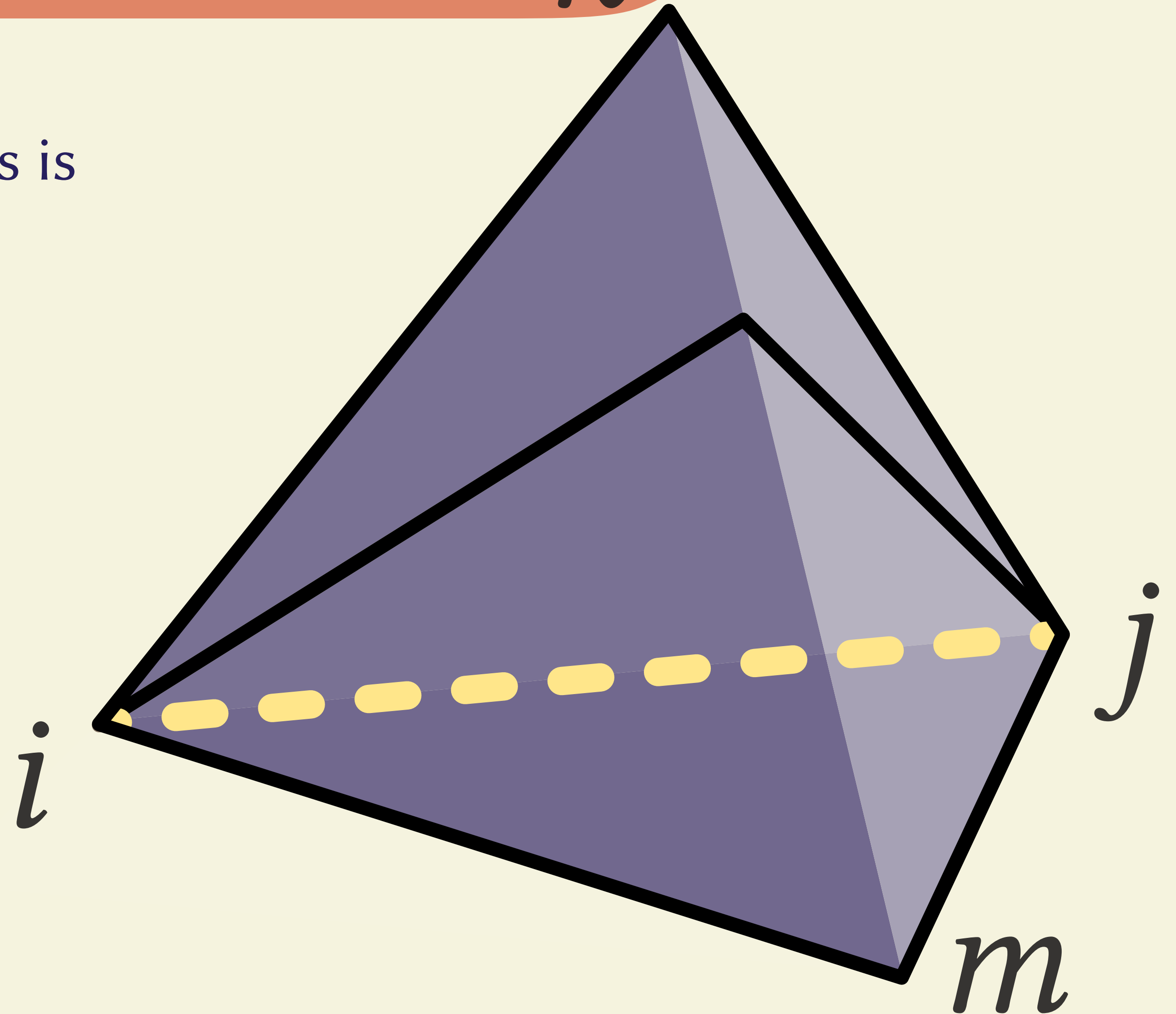
- Can't immediately tell which edge this is



Normal coordinates are not enough to encode correspondence

 k

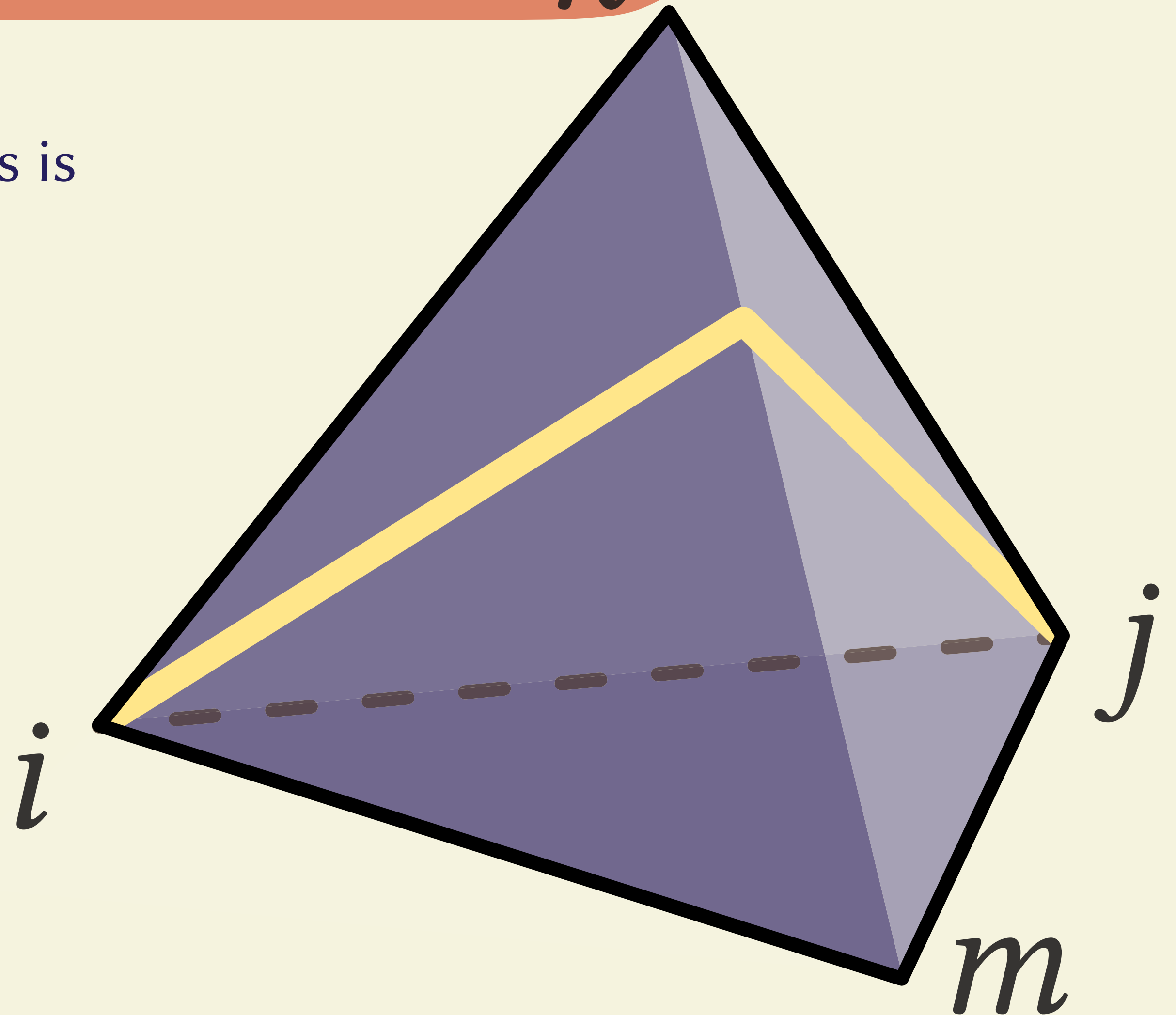
- Can't immediately tell which edge this is



Normal coordinates are not enough to encode correspondence

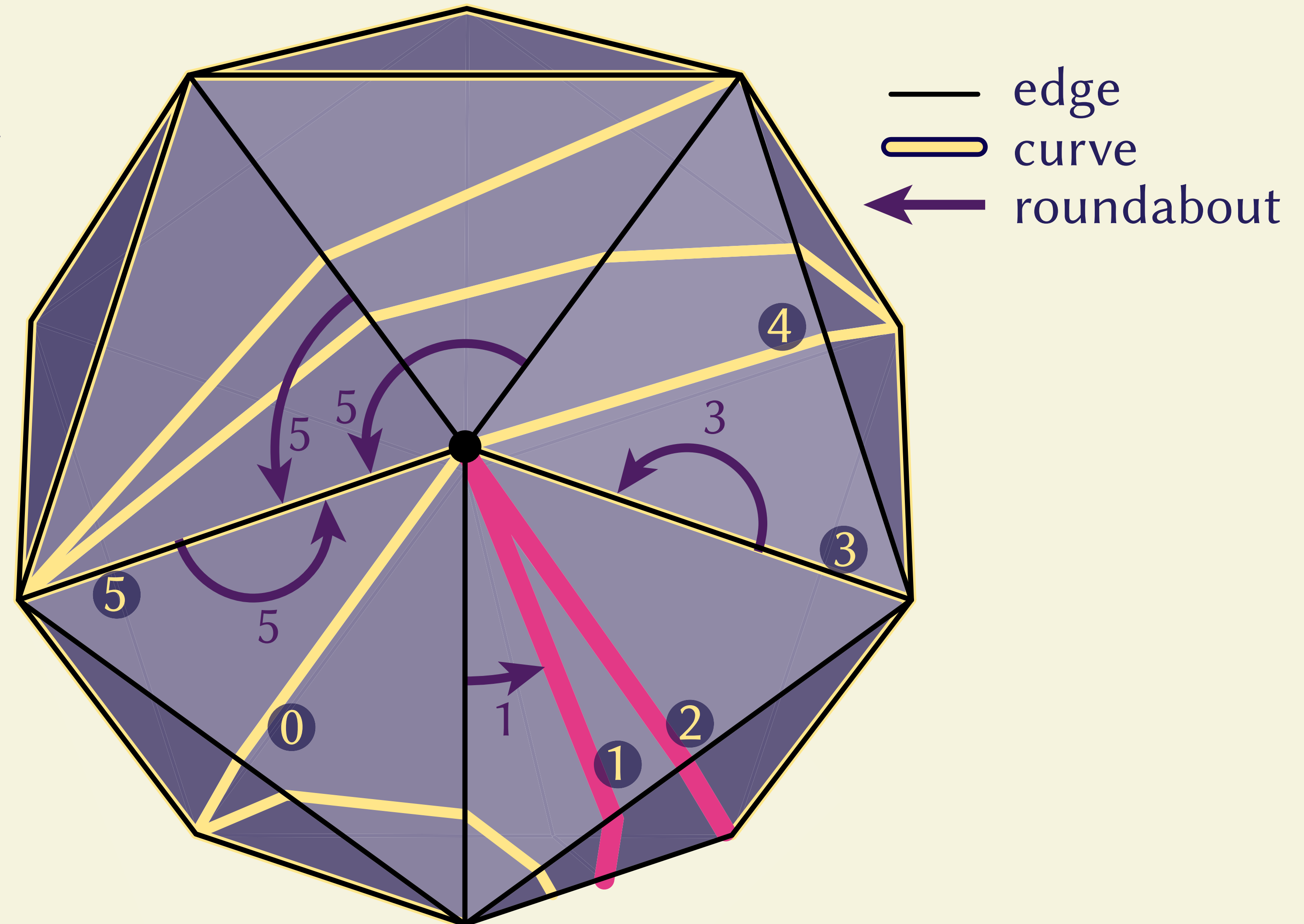
 k

- Can't immediately tell which edge this is
 - Disambiguate with roundabouts

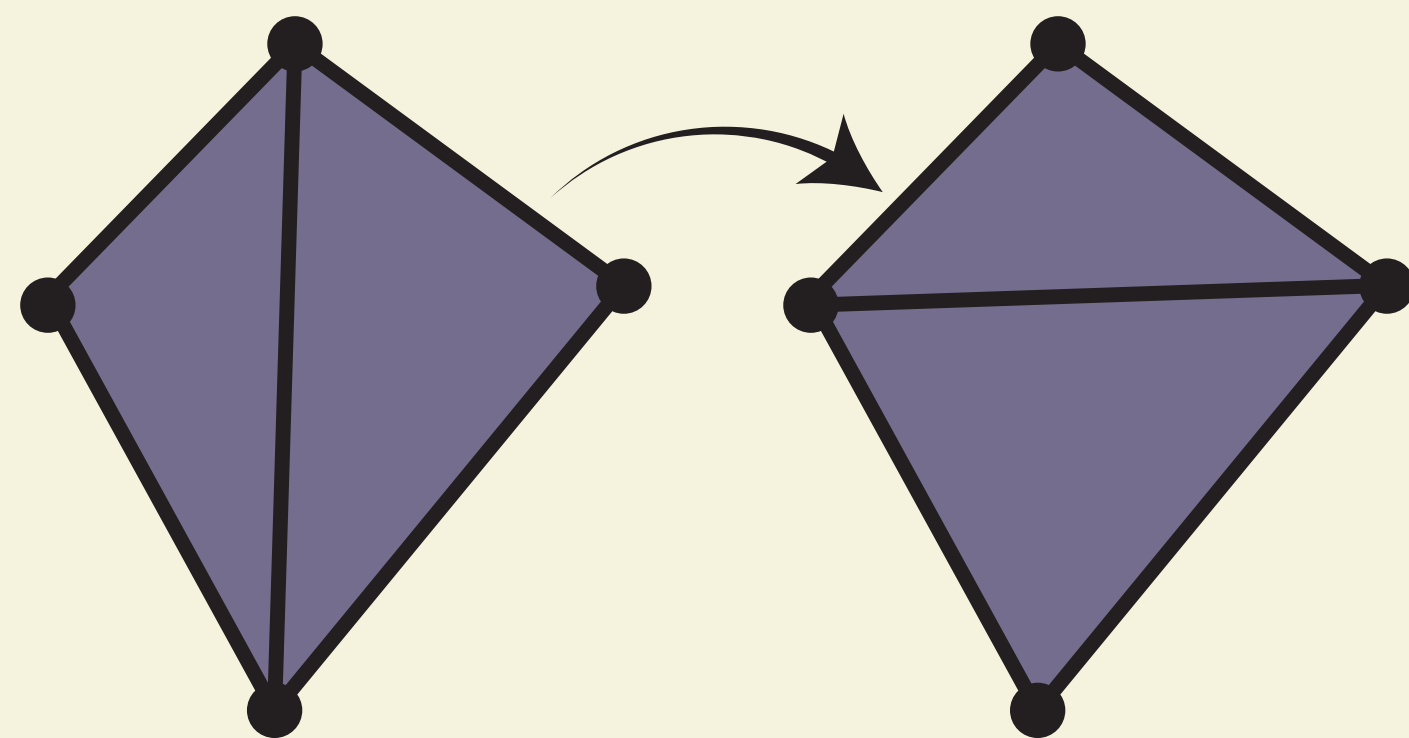
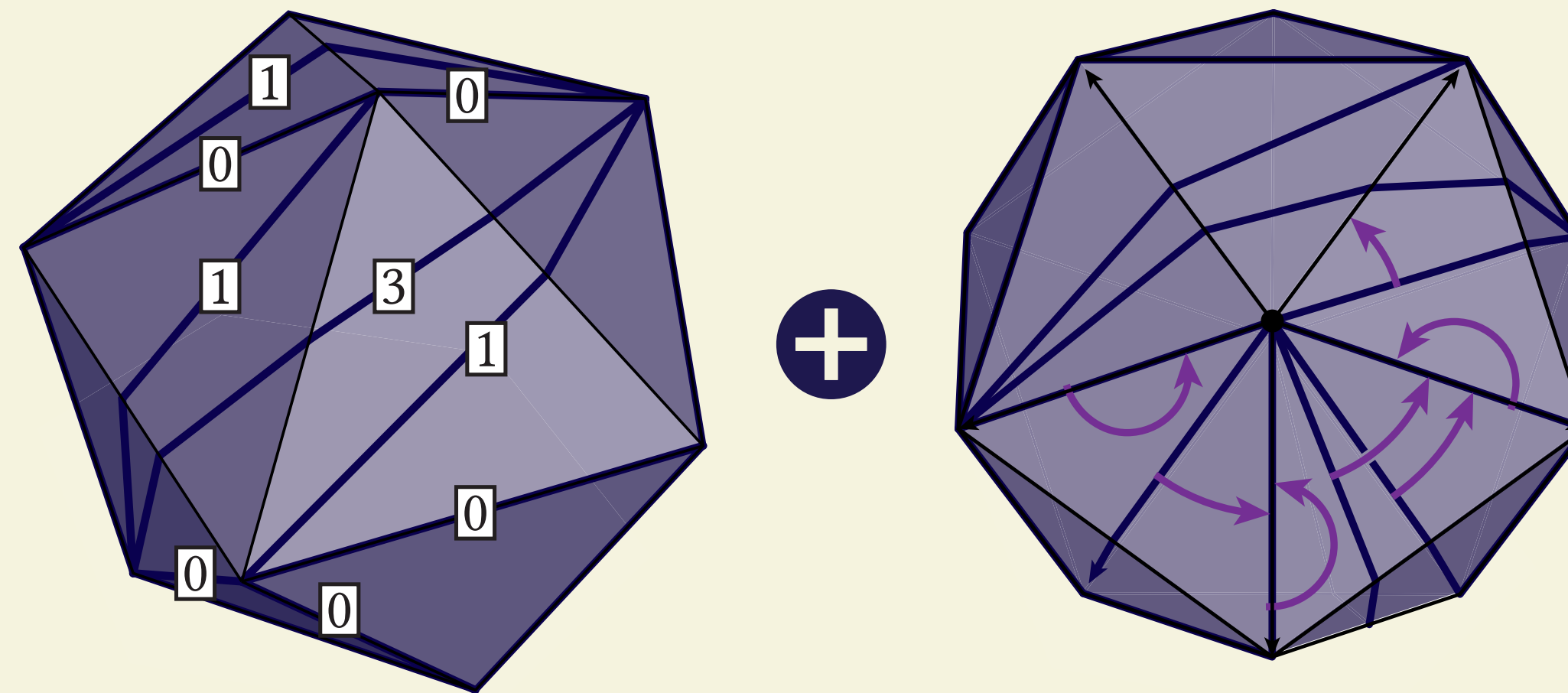


Roundabouts

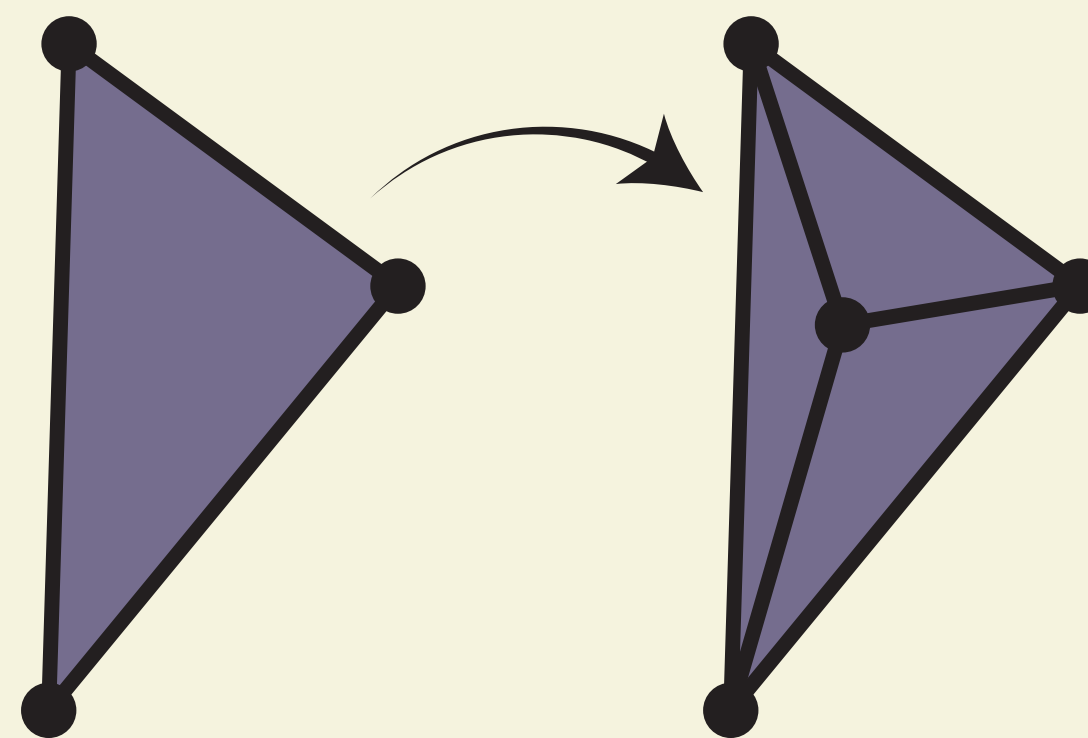
- Each black edge stores a pointer to the next yellow curve
- Resolves all ambiguity



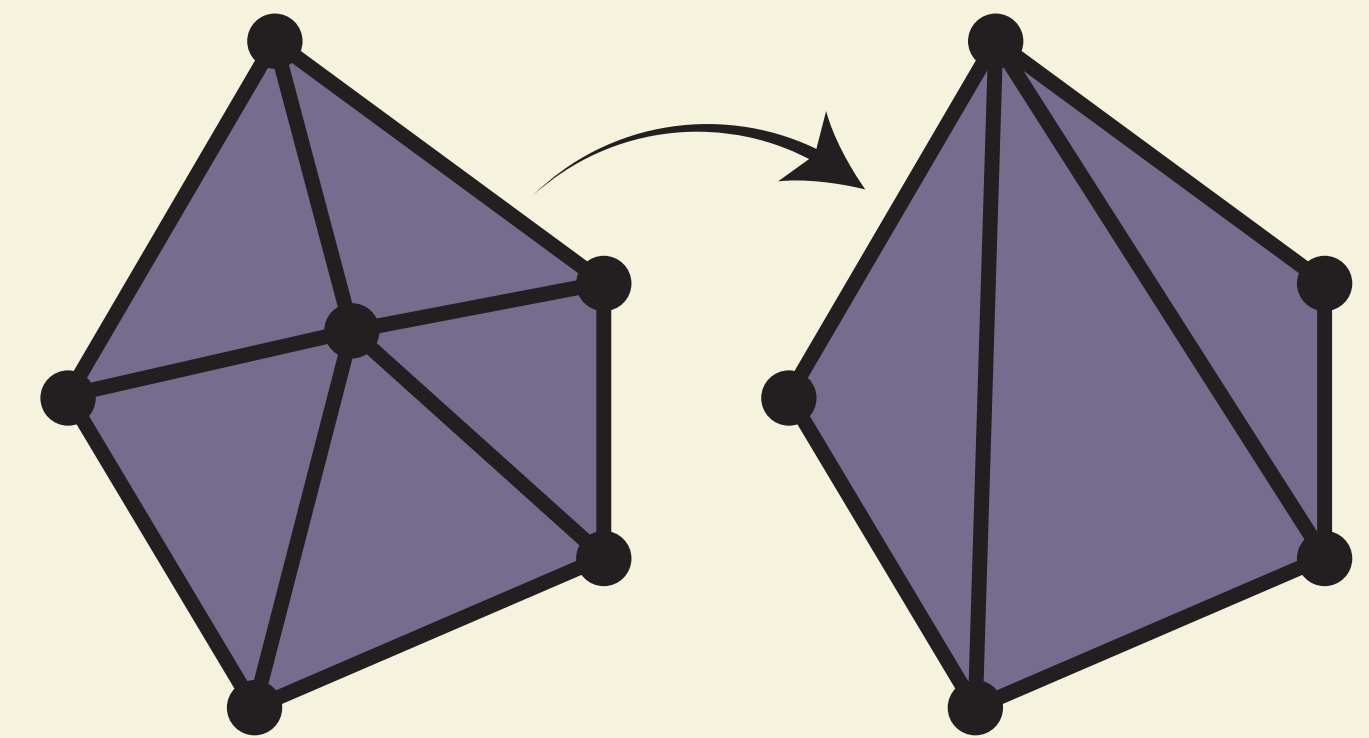
Data structure operations



edge flips

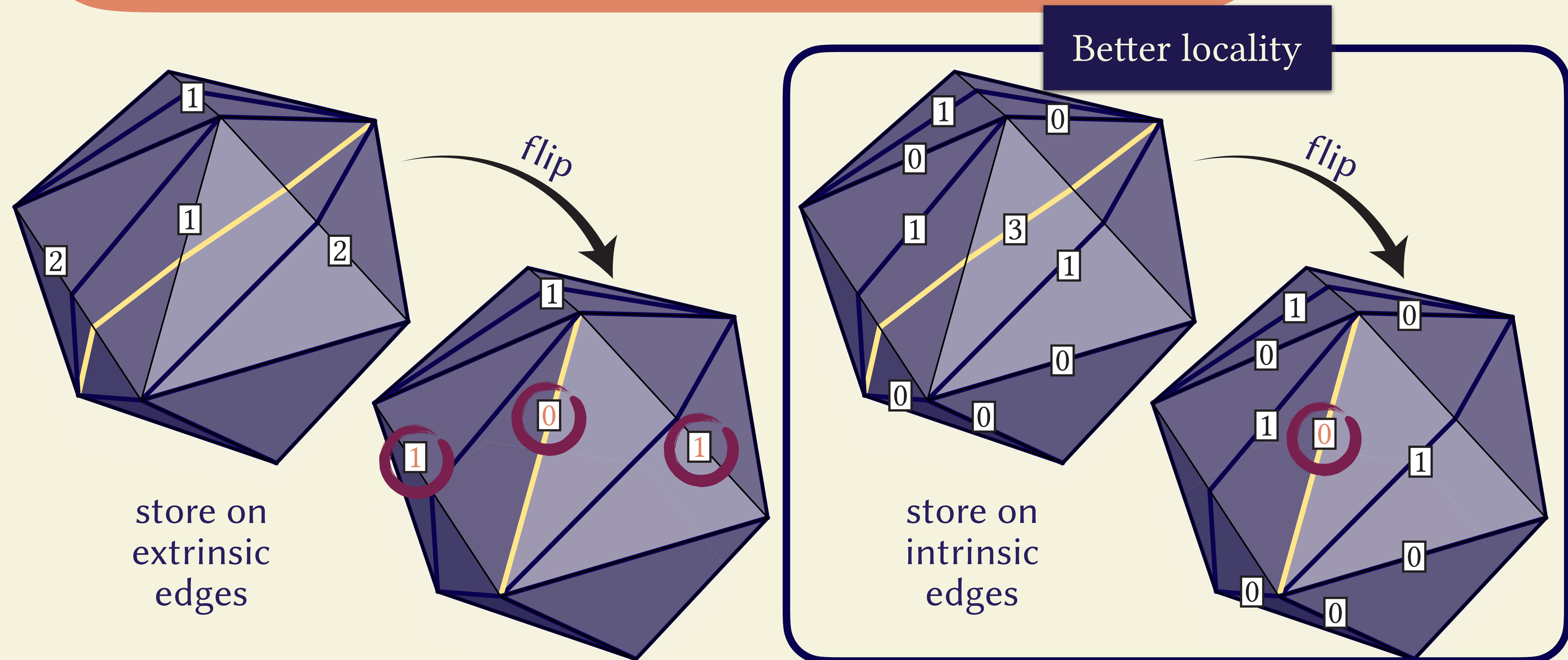


vertex insertion



flat vertex removal

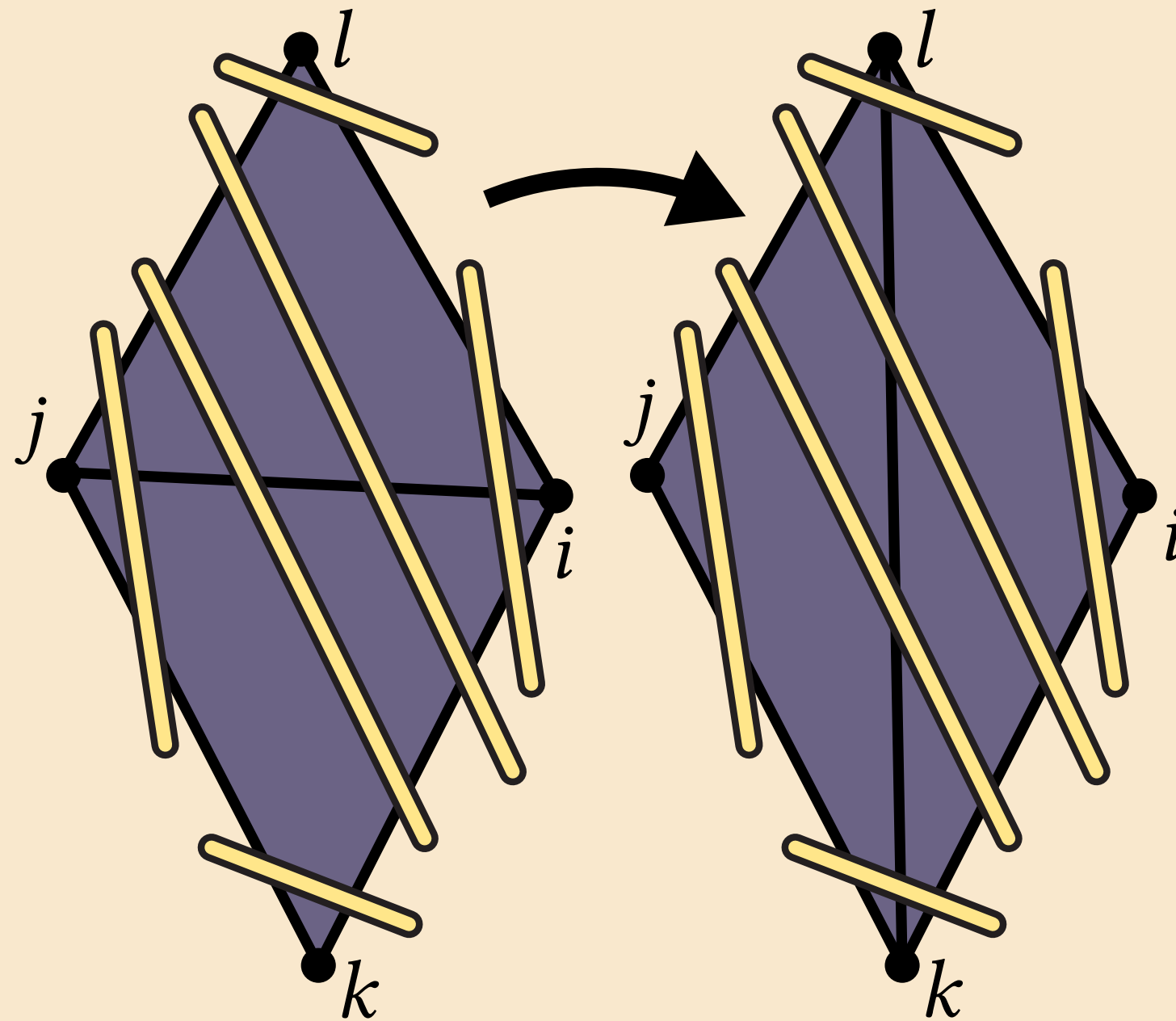
Where to store integer coordinates



Edge flips

— edge
 ═ curve

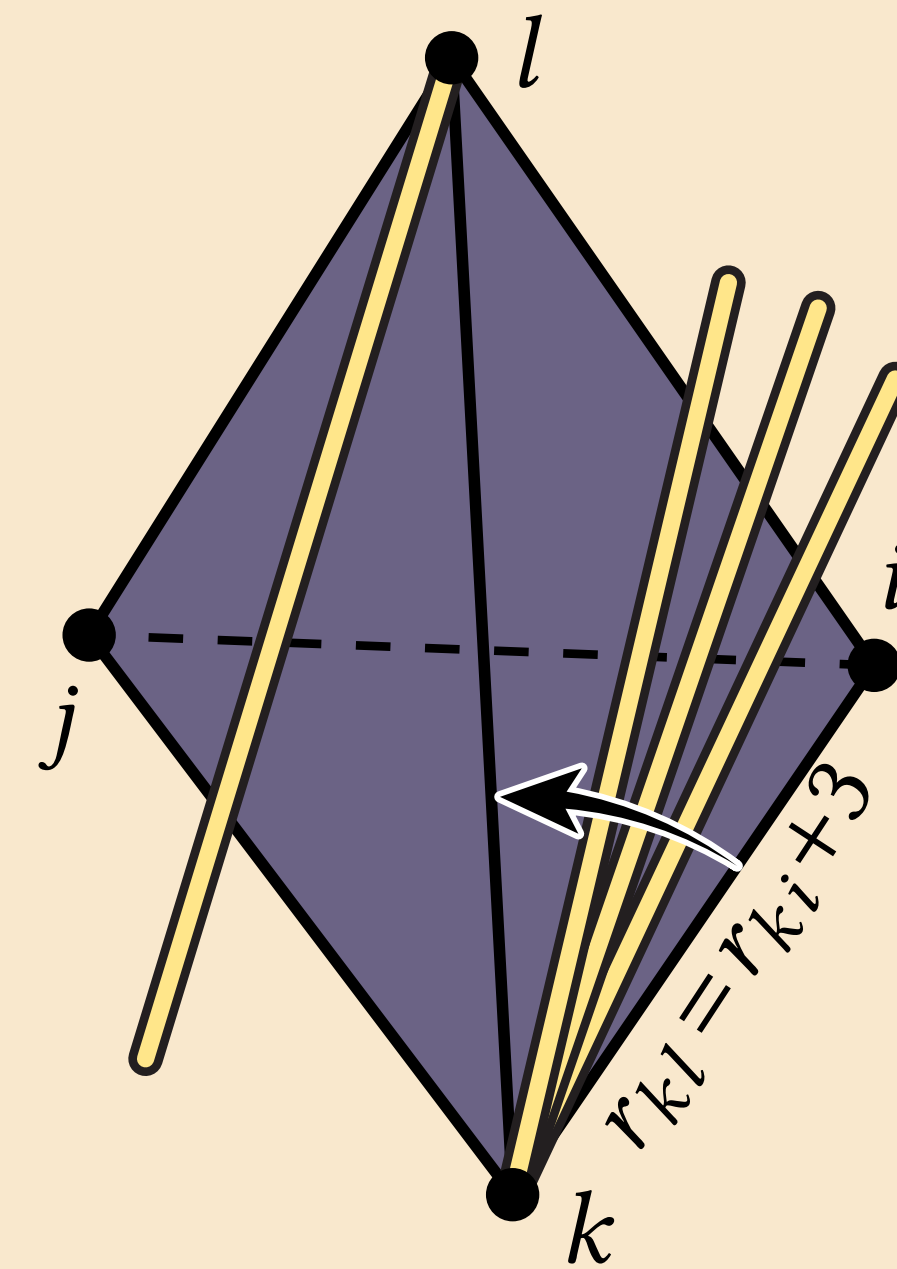
Normal Coordinates



$$n_{kl} = \max(n_{ki} + n_{lj}, n_{jk} + n_{li}) - n_{ij}$$

(if there are no endpoints)

Roundabouts

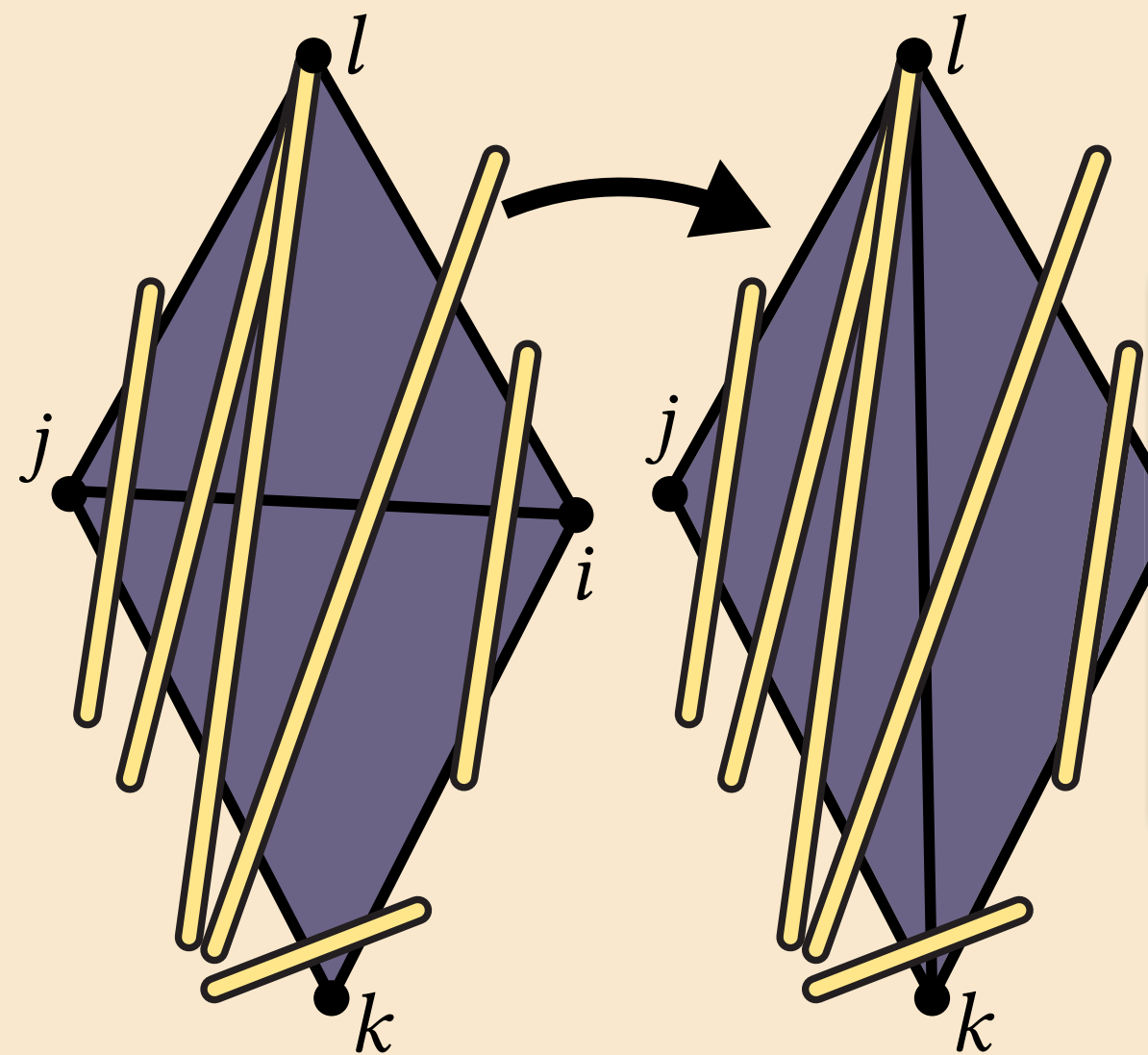


$$r_{kl} = r_{ki} + n_{ki}^- + \max(0, n_{il}^+ - n_{lk}^+ - n_{ki}^+)$$

Edge flips

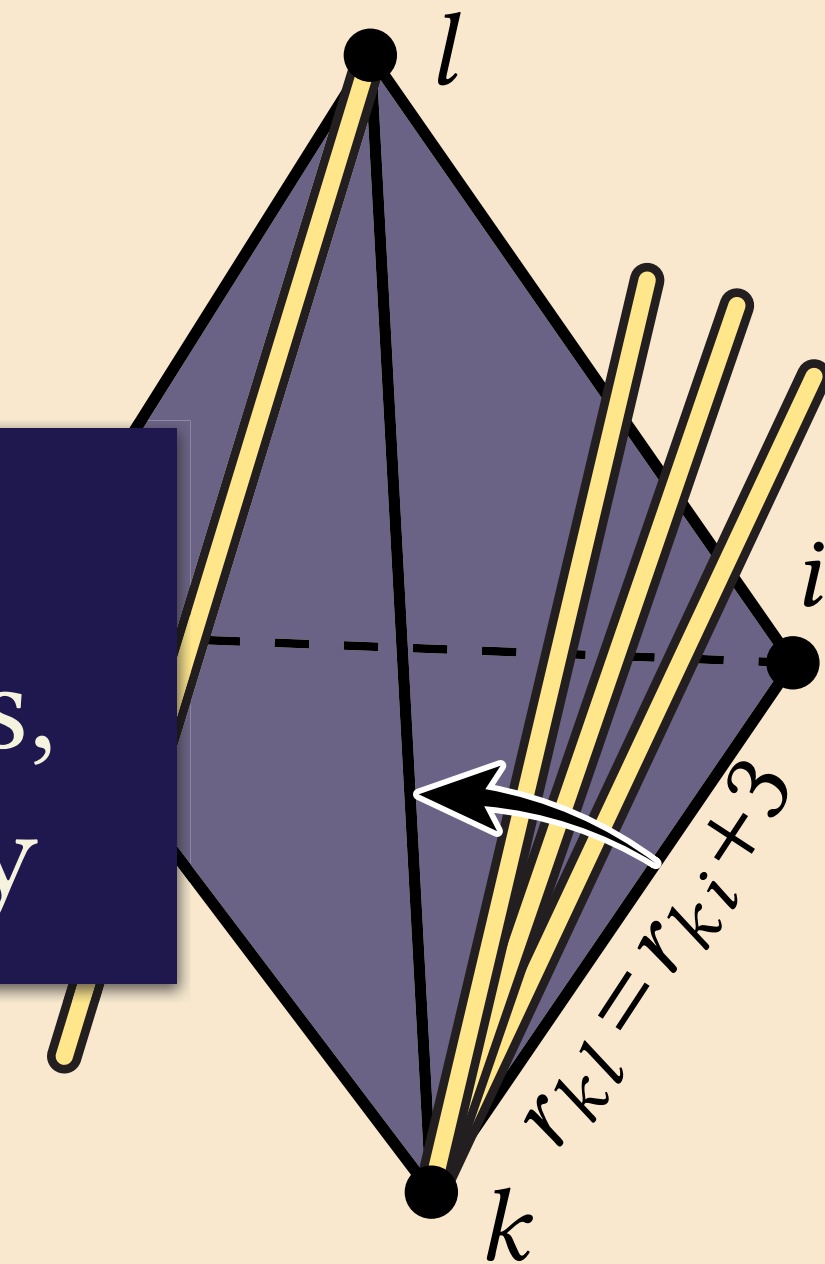
— edge
 ═ curve

Normal Coordinates



Key takeaway:
 closed form flip formulas,
 independent of geometry

Roundabouts



$$n_{kl} = c_l^{jk} + c_k^{ij} + \frac{1}{2} \left| c_j^{il} - c_j^{ki} \right| + \frac{1}{2} \left| c_i^{lj} - c_i^{jk} \right| - \frac{1}{2} e_l^{ji} - \frac{1}{2} e_k^{ij} + e_i^{lj} + e_i^{jk} + e_j^{il} + e_j^{ki} + n_{ij}^-$$

(general case)

where

$$2c_k^{ij} := \max(0, n_{jk}^+ + n_{ki}^+ - n_{ij}^+ - e_i^{jk} - e_j^{ki})$$

$$e_k^{ij} := \max(0, n_{ij}^+ - n_{jk}^+ - n_{ki}^+)$$

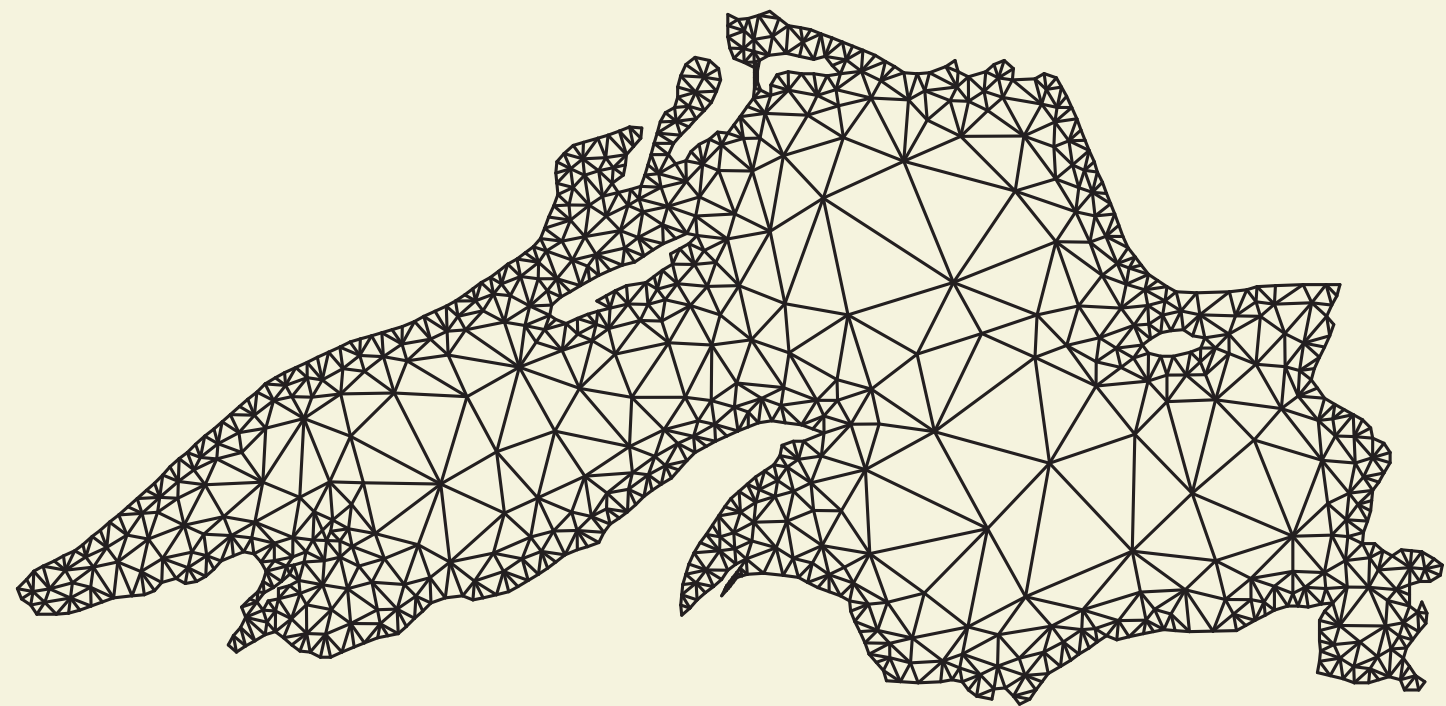
$$r_{kl} = r_{ki} + n_{ki}^- + \max(0, n_{il}^+ - n_{lk}^+ - n_{ki}^+)$$

Applications

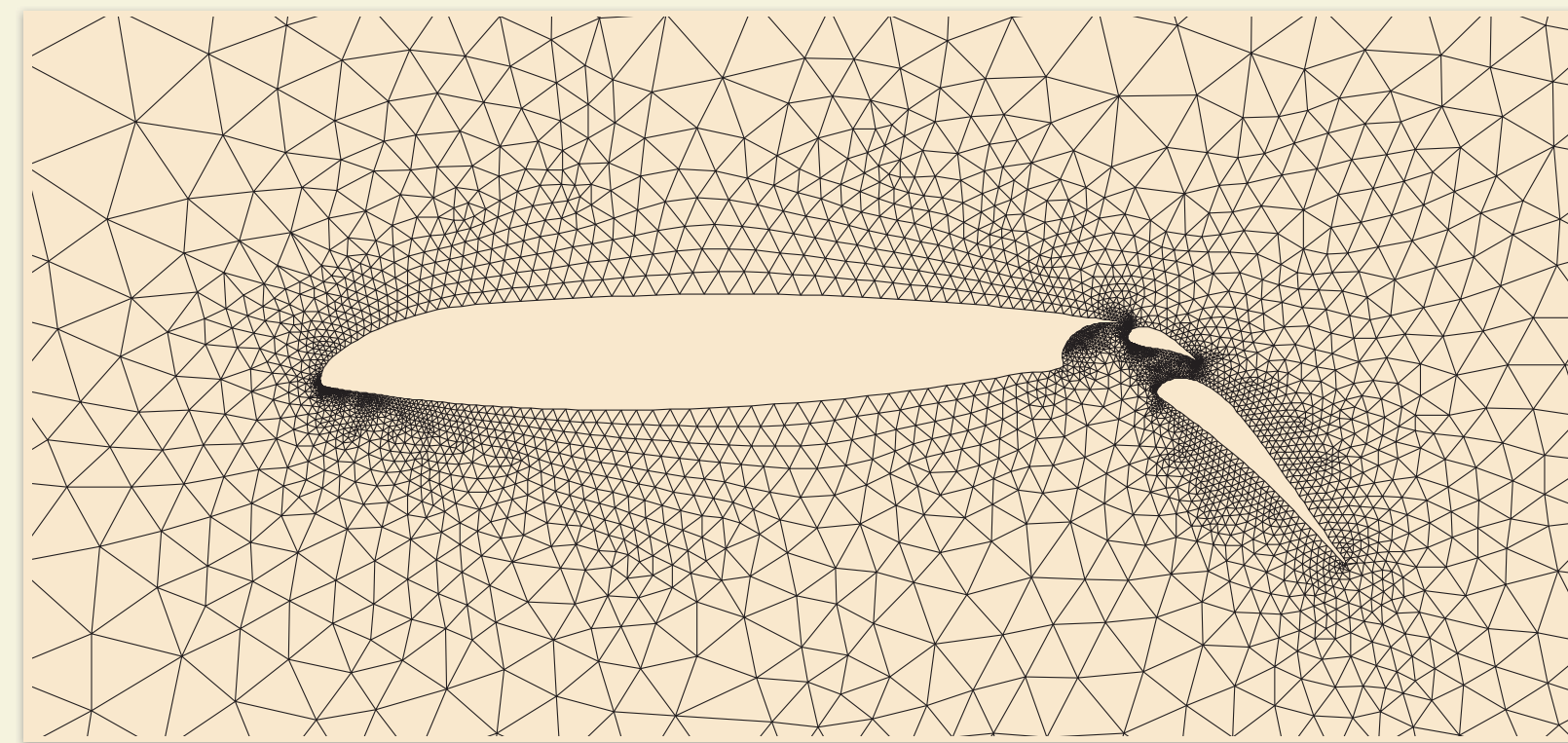


Delaunay refinement for planar meshing

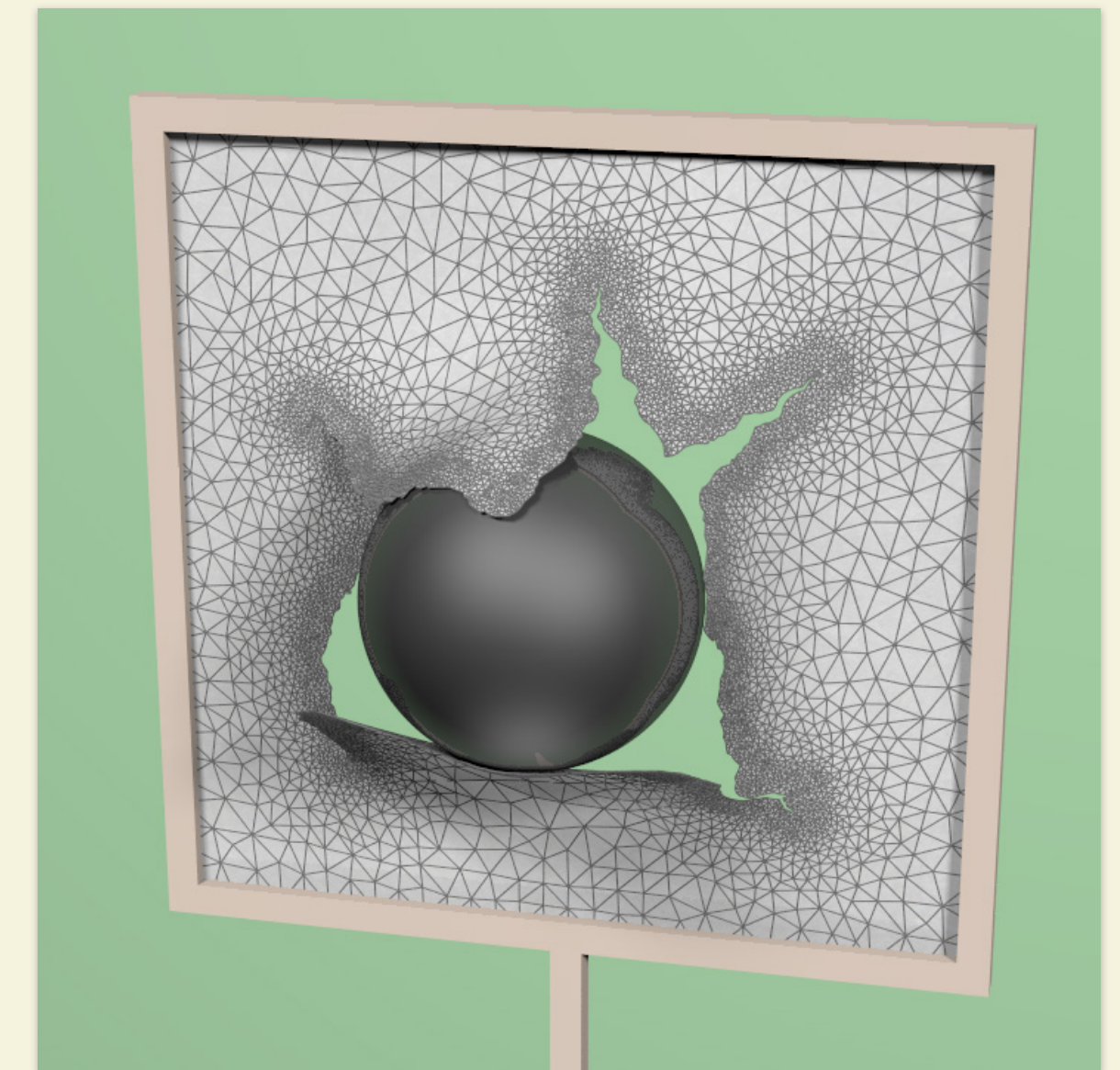
- Crucial tool in 2D - remesh with guaranteed quality bounds [Chew 1993; Shewchuk 1997]



[Shewchuk 1997]



[Shewchuk 1997]



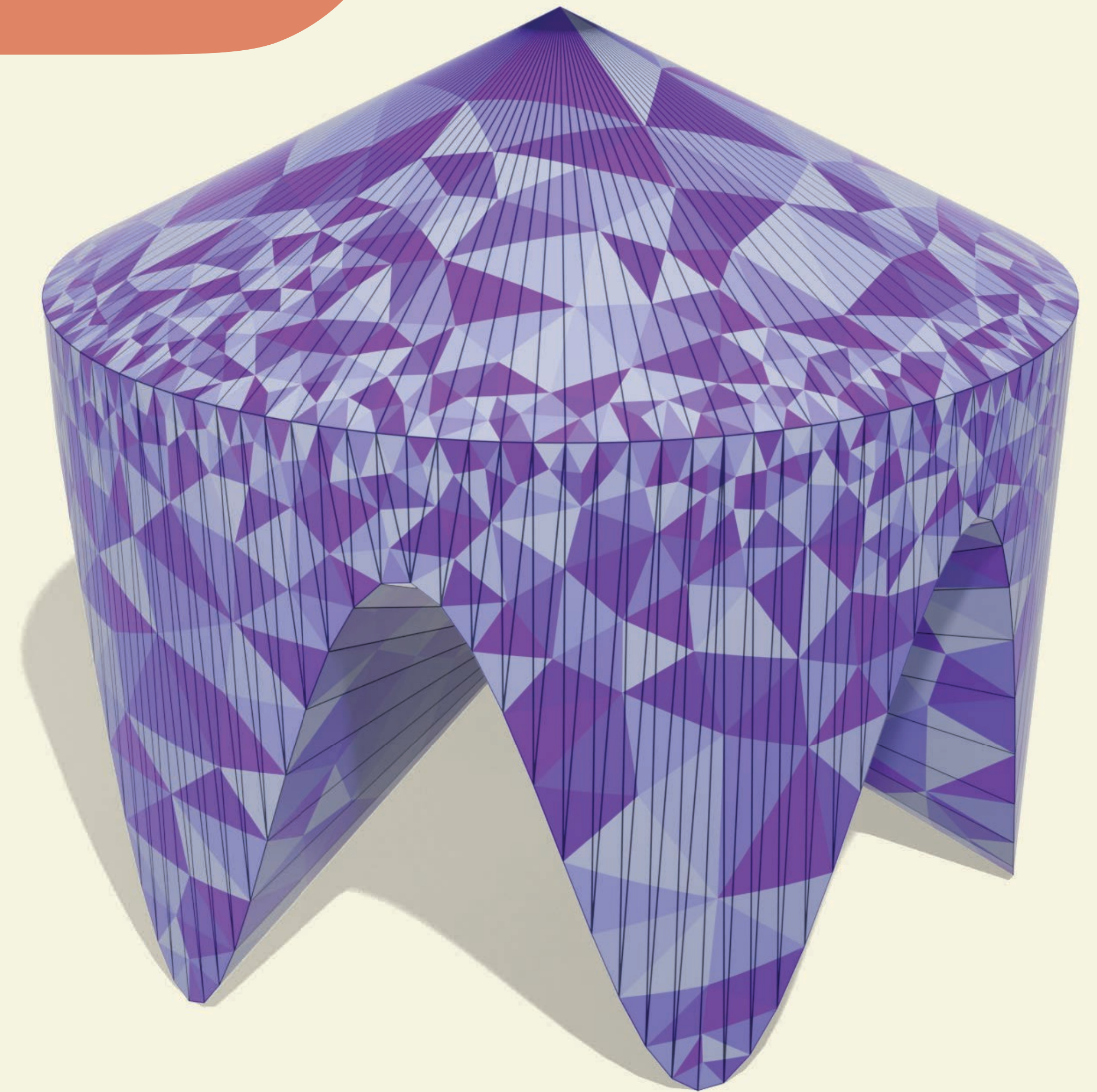
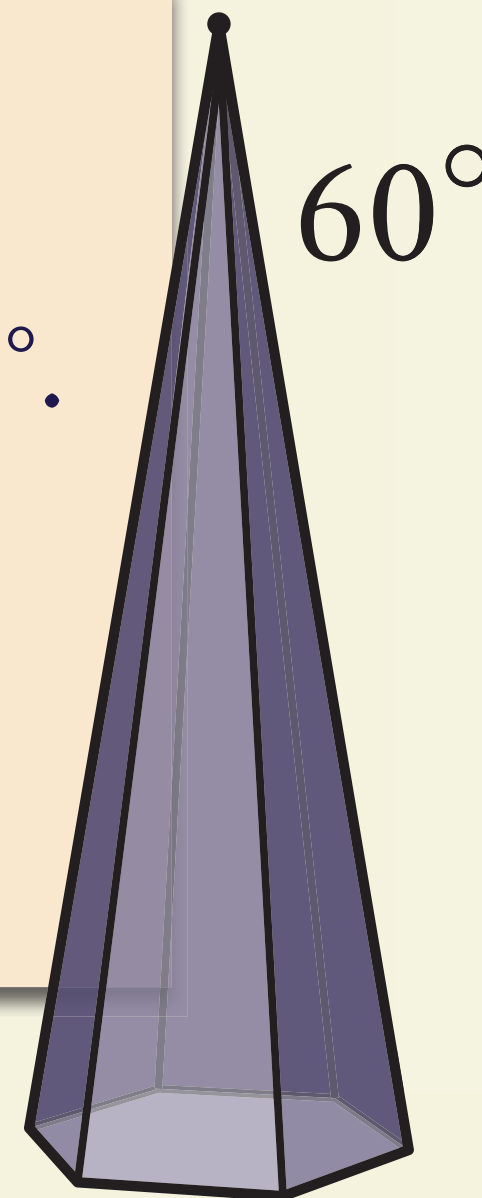
[Busaryev+ 2013]

Intrinsic Delaunay refinement

- Intrinsic retriangulation algorithm proposed by [Sharp, Soliman & Crane 2019]

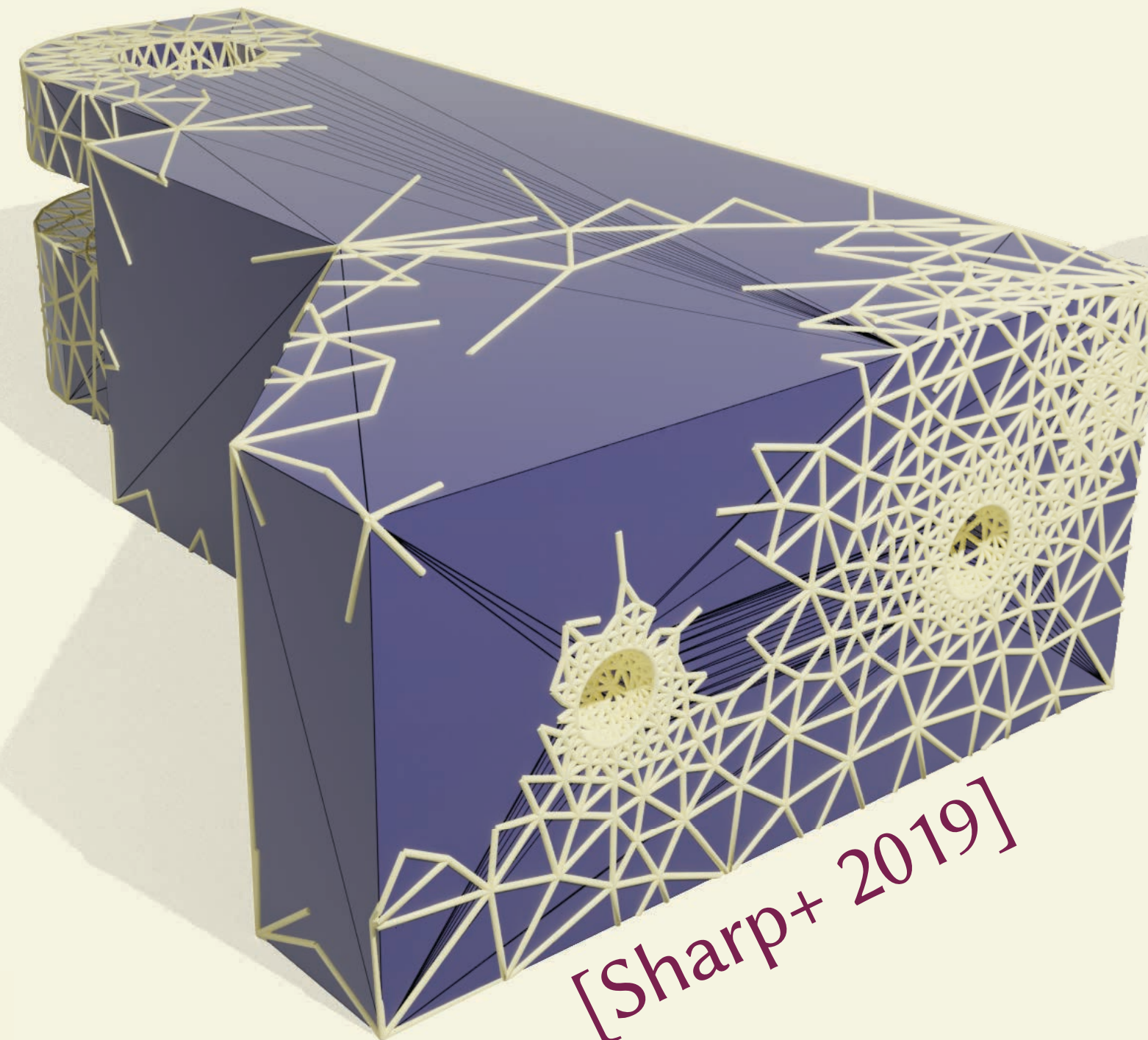
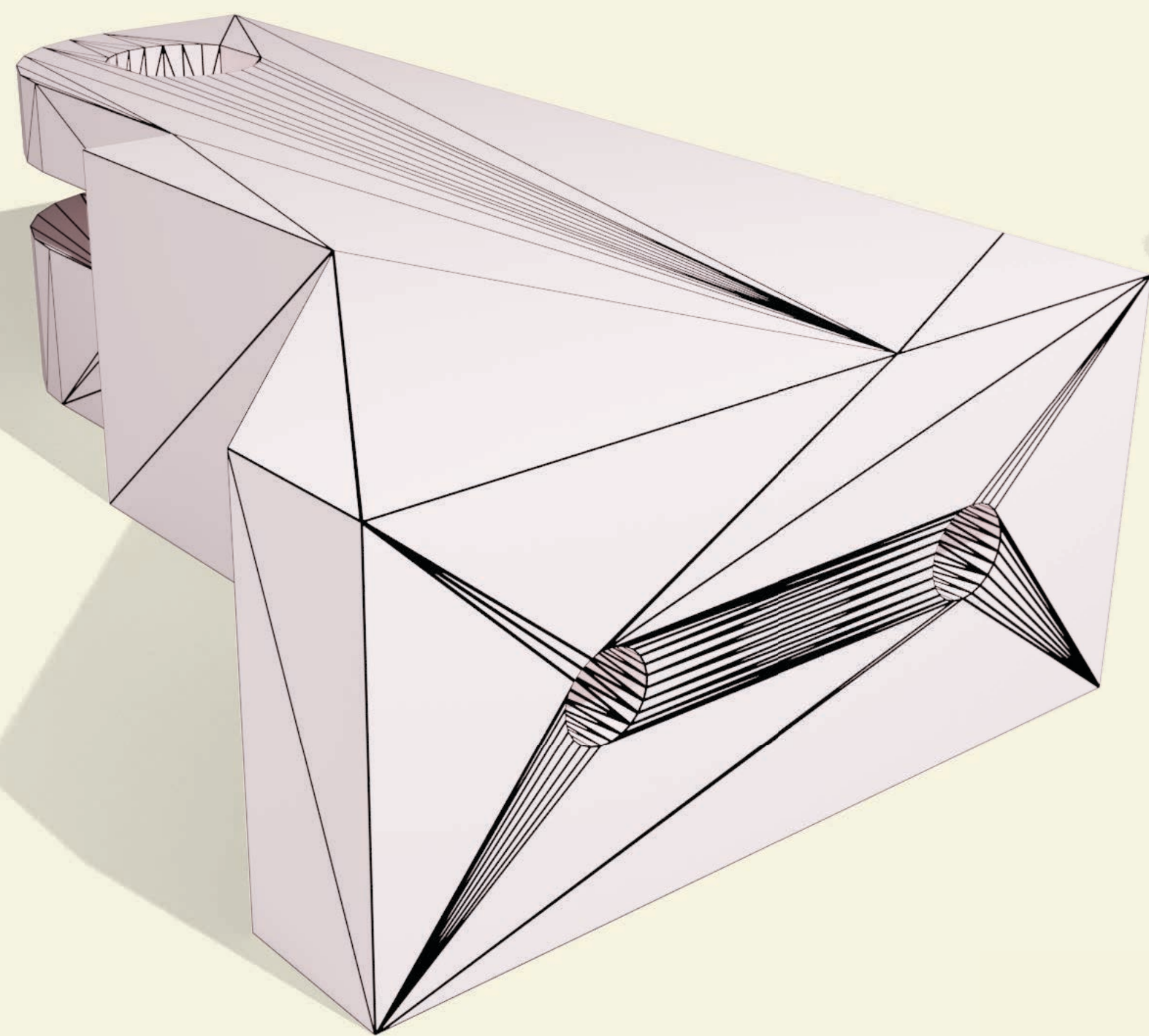
Theorem [G., Sharp & Crane 2021]

Let M be a mesh without boundary whose cone angles are all at least 60° . Then intrinsic Delaunay refinement produces a Delaunay mesh with triangle corner angles at least 30°

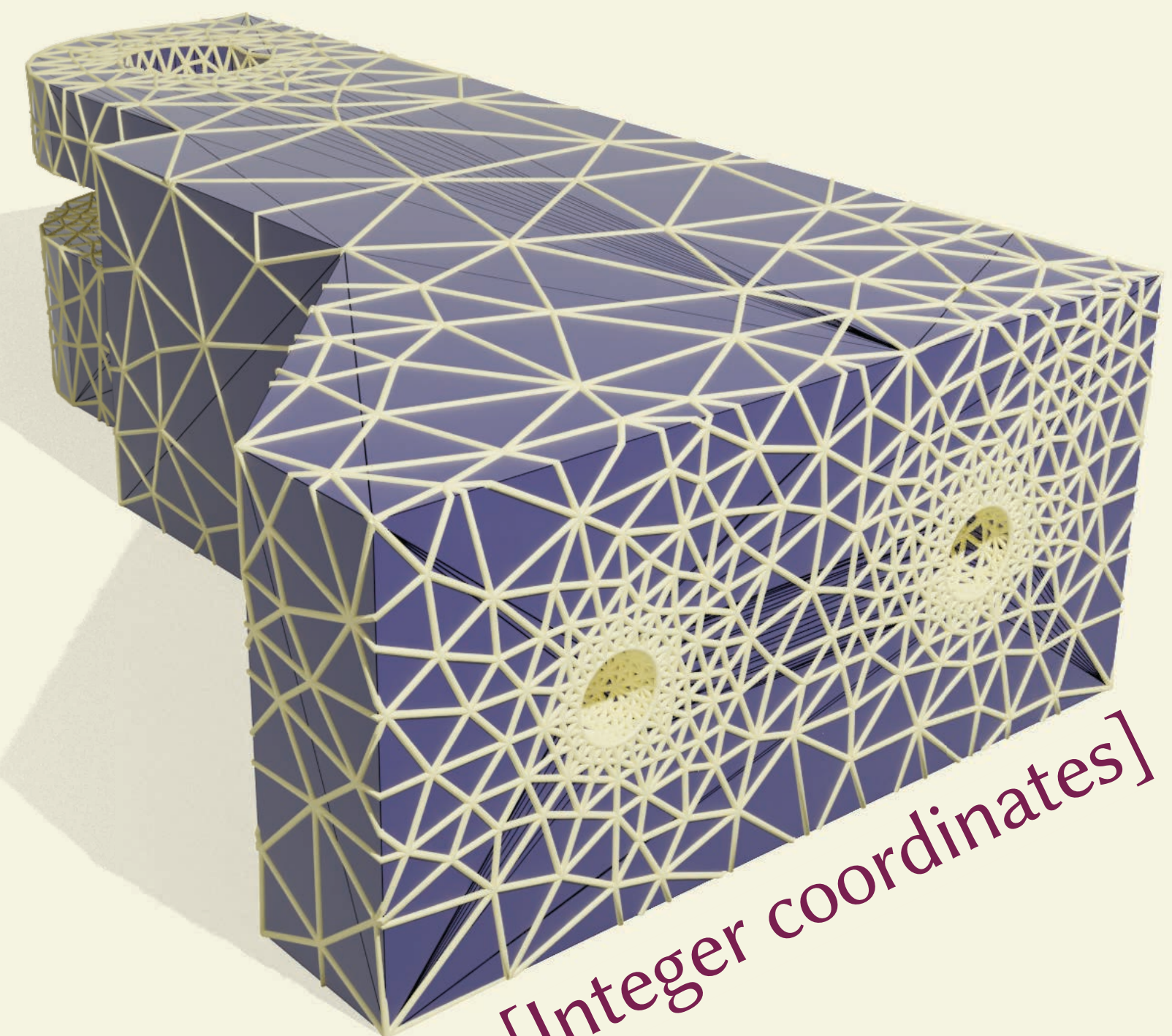


Common subdivisions of intrinsic Delaunay refinements

- Integer coordinates can be crucial to recovering the common subdivision



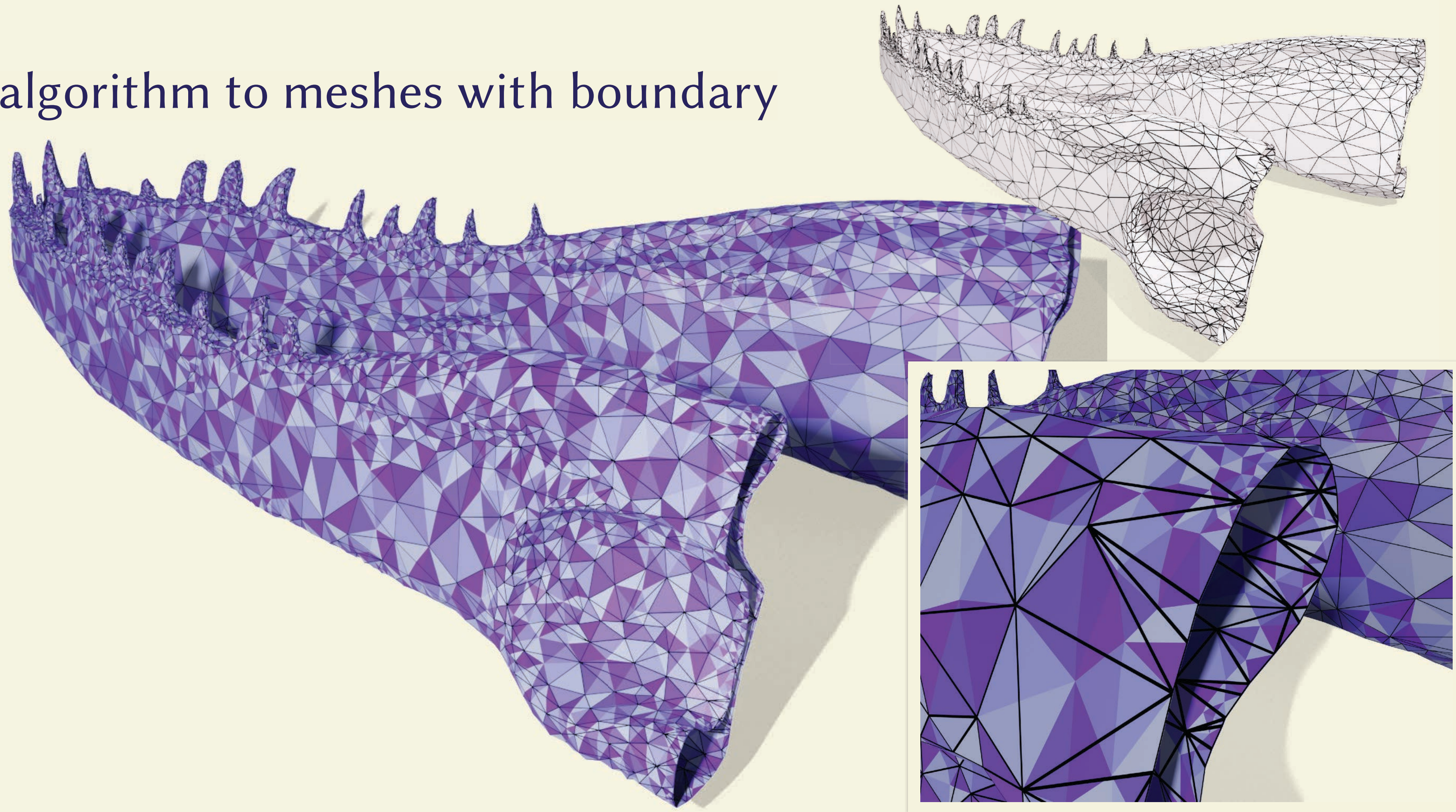
[Sharp+ 2019]



[Integer coordinates]

Intrinsic Delaunay refinement of meshes with boundary

- Extend algorithm to meshes with boundary

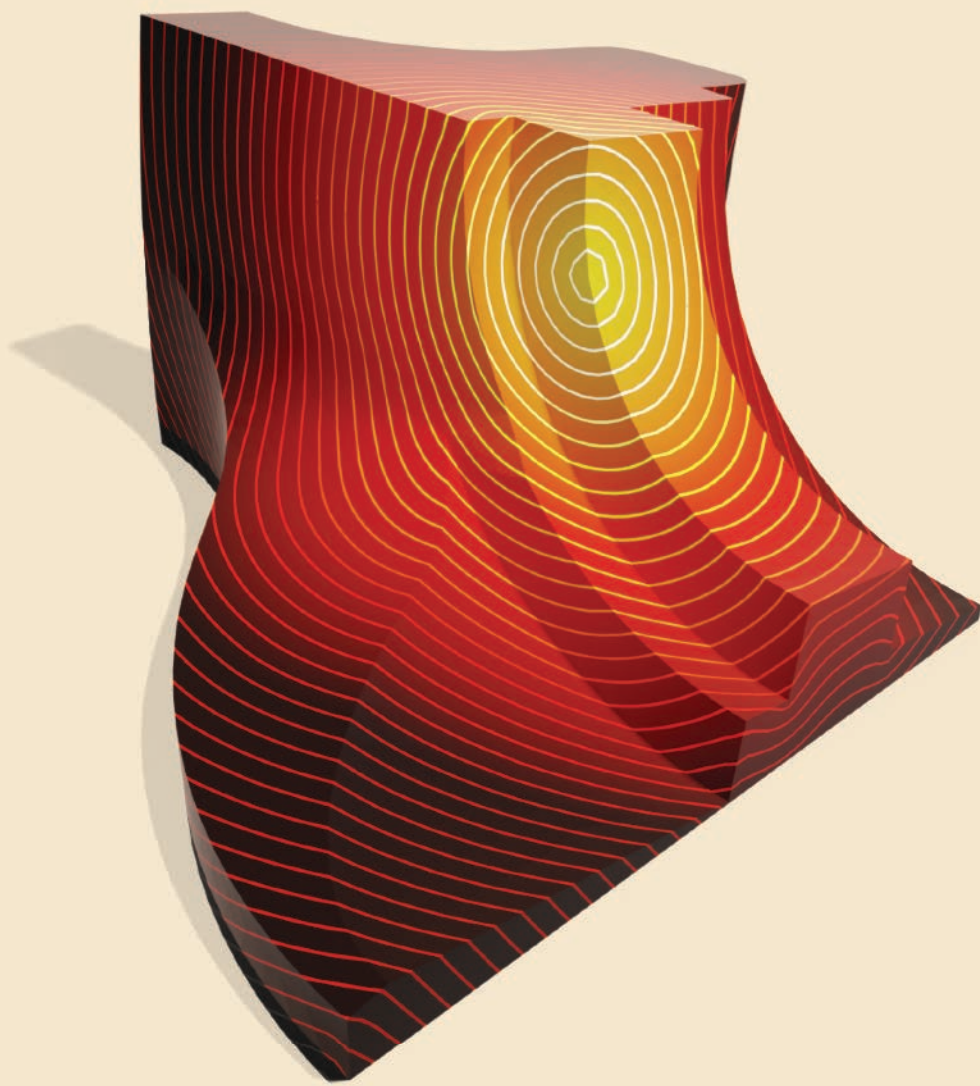


Intrinsic Delaunay refinement — validation

- Compute refinements & common subdivisions for Thingi10k dataset [Zhou & Jacobson 2016]
 - 7696 manifold meshes
- $< 1s$ on most meshes; only took $> 1m$ on 6 meshes
- 100% success rate for refinement & common subdivision
 - [Sharp, Soliman & Crane 2019] succeed on only 69.1% of meshes



Application: PDE-Based Geometry Processing



heat method for
geodesic distance
[Crane, Weischedel
& Wardetzky 2013]



mean error: 28%
result on input
mesh



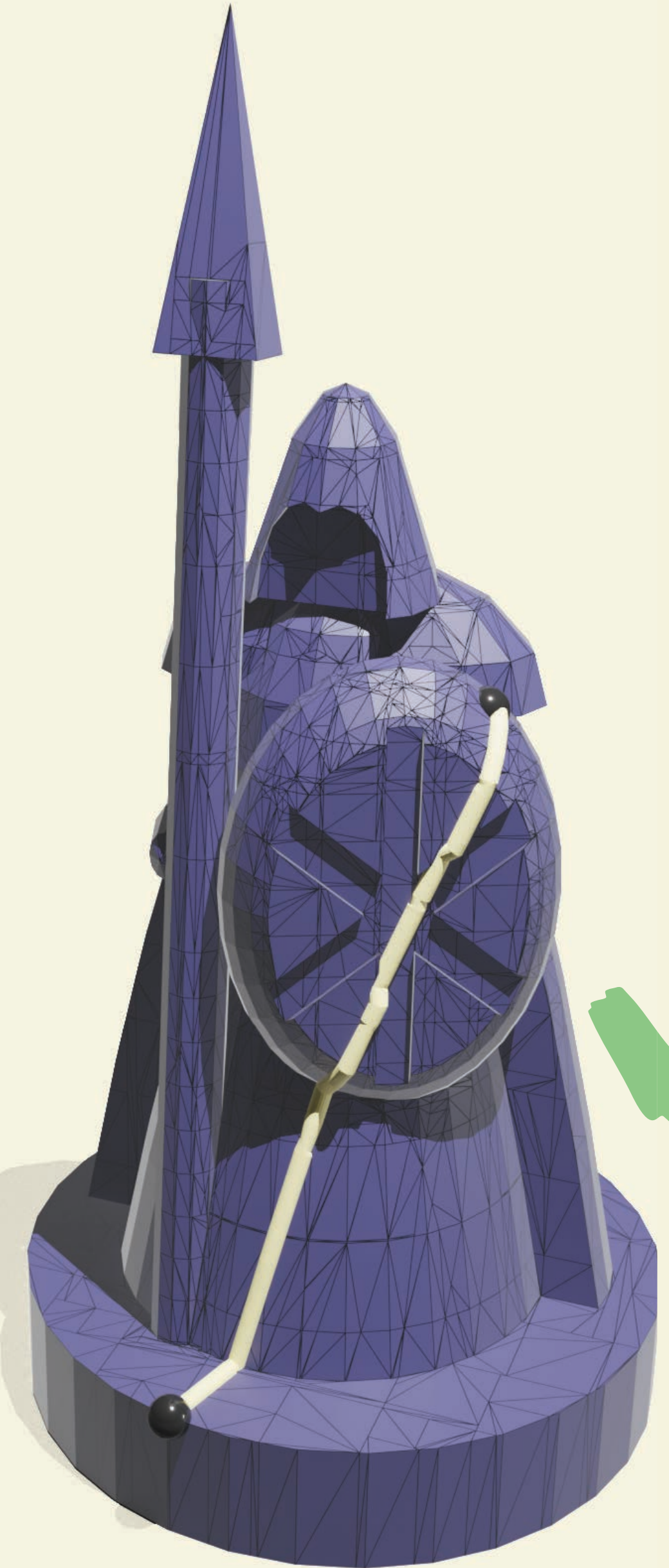
mean error: 2%
result on
Delaunay
refinement

Application: Flip-Based Geodesic Paths

- FlipOut [Sharp & Crane 2020]:
 - computes geodesic paths via edge flips

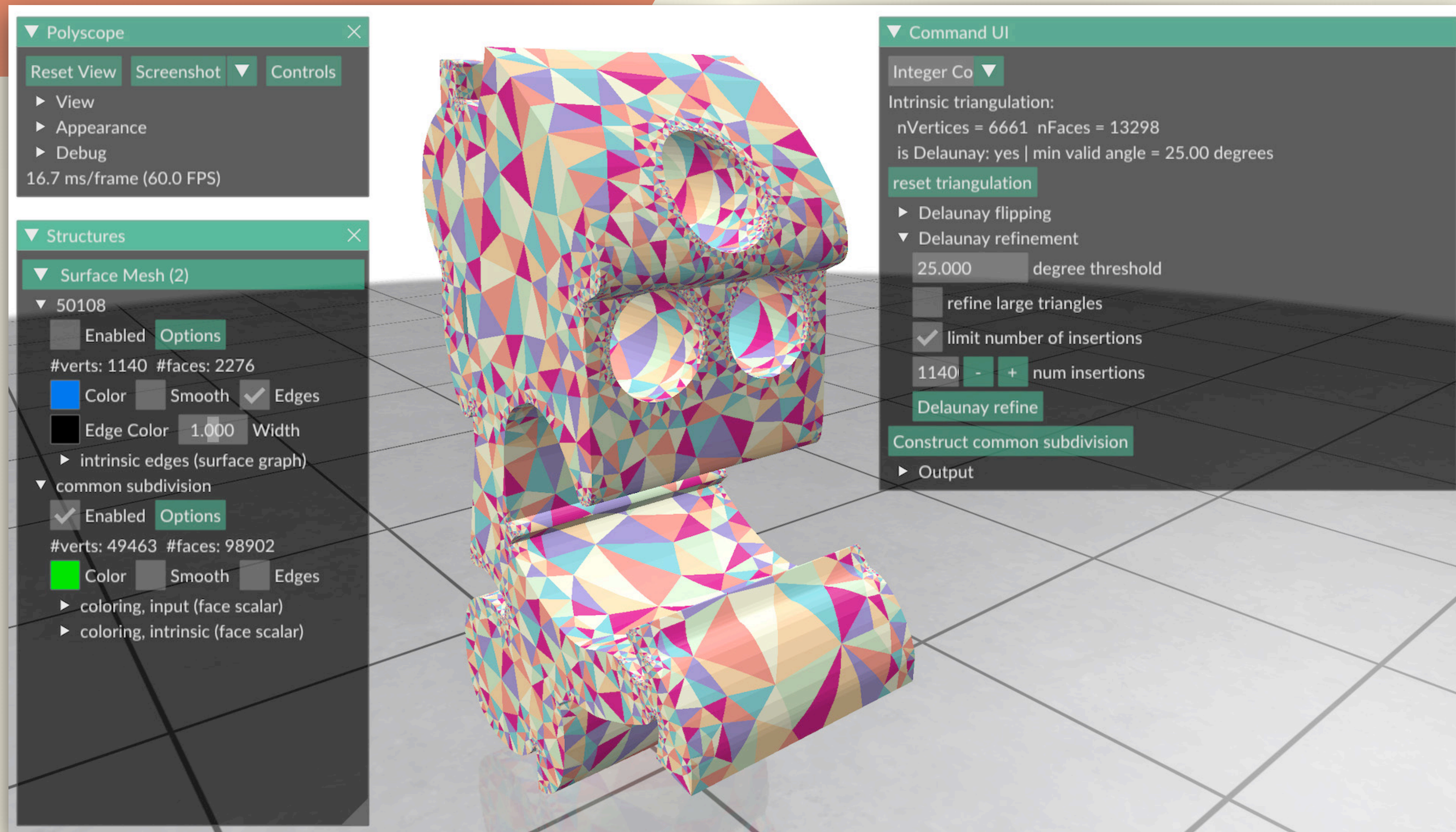


[Sharp, Soliman
& Crane 2019]



[Integer coordinates]

Try it out yourself

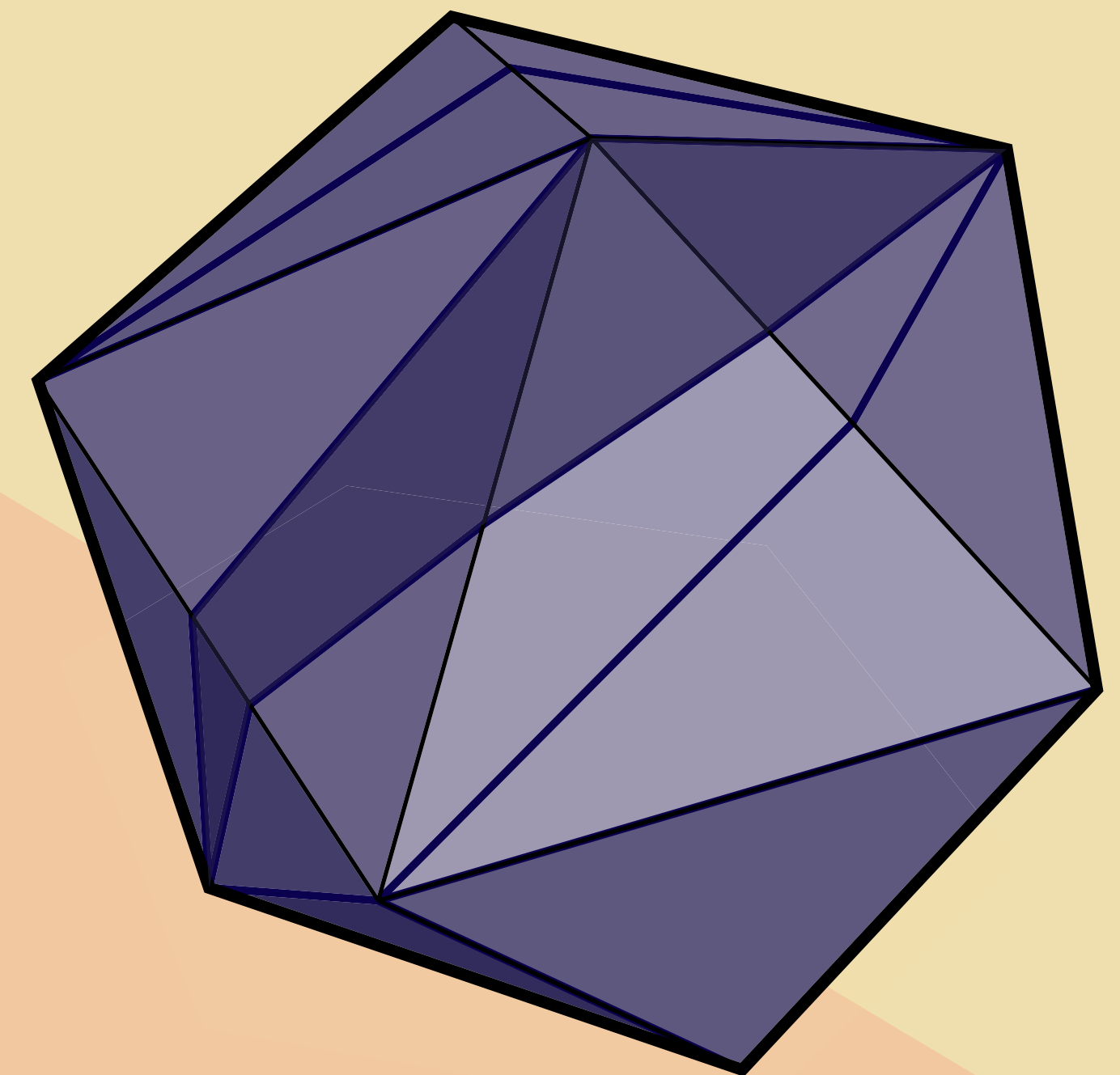
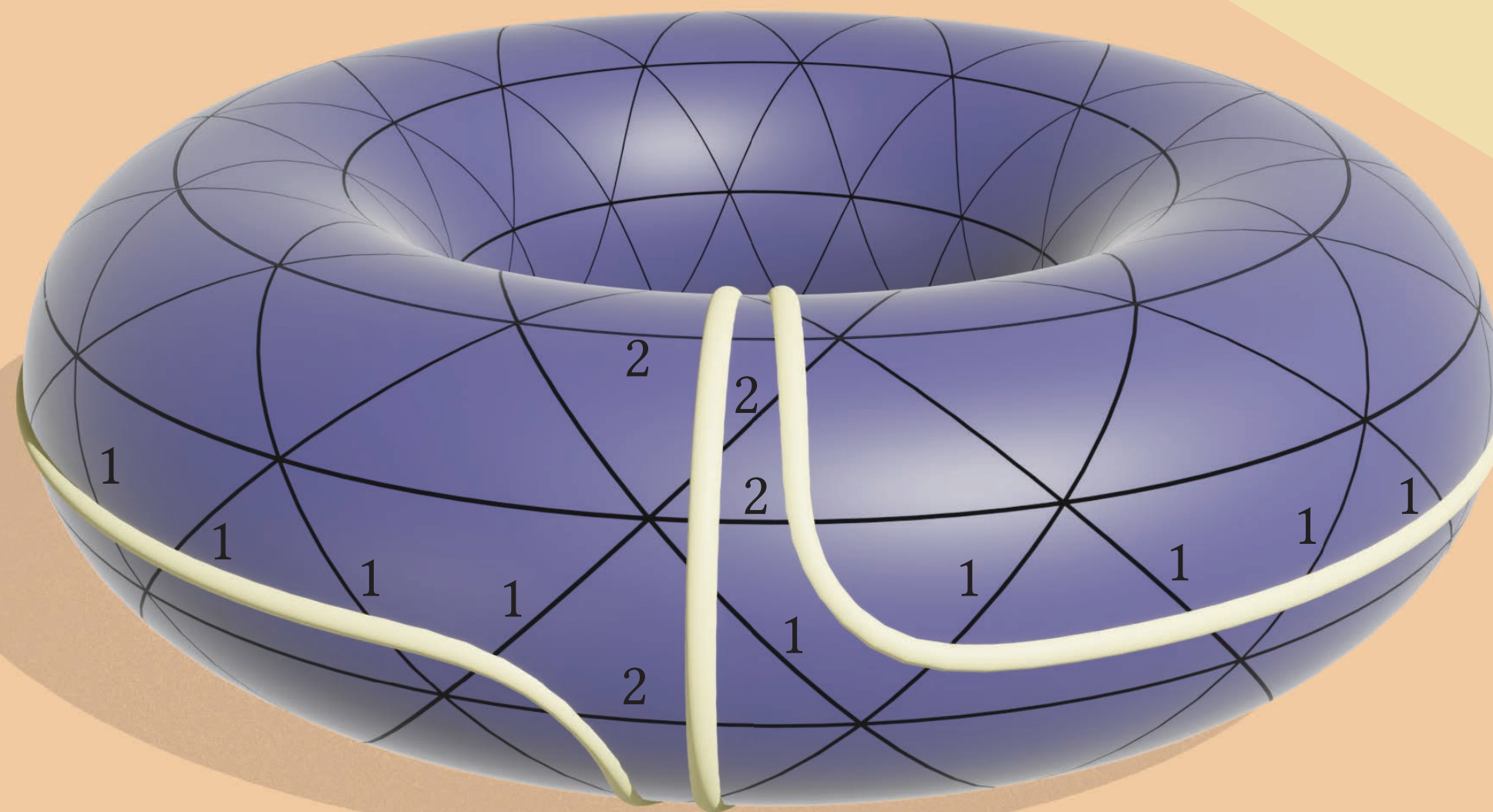
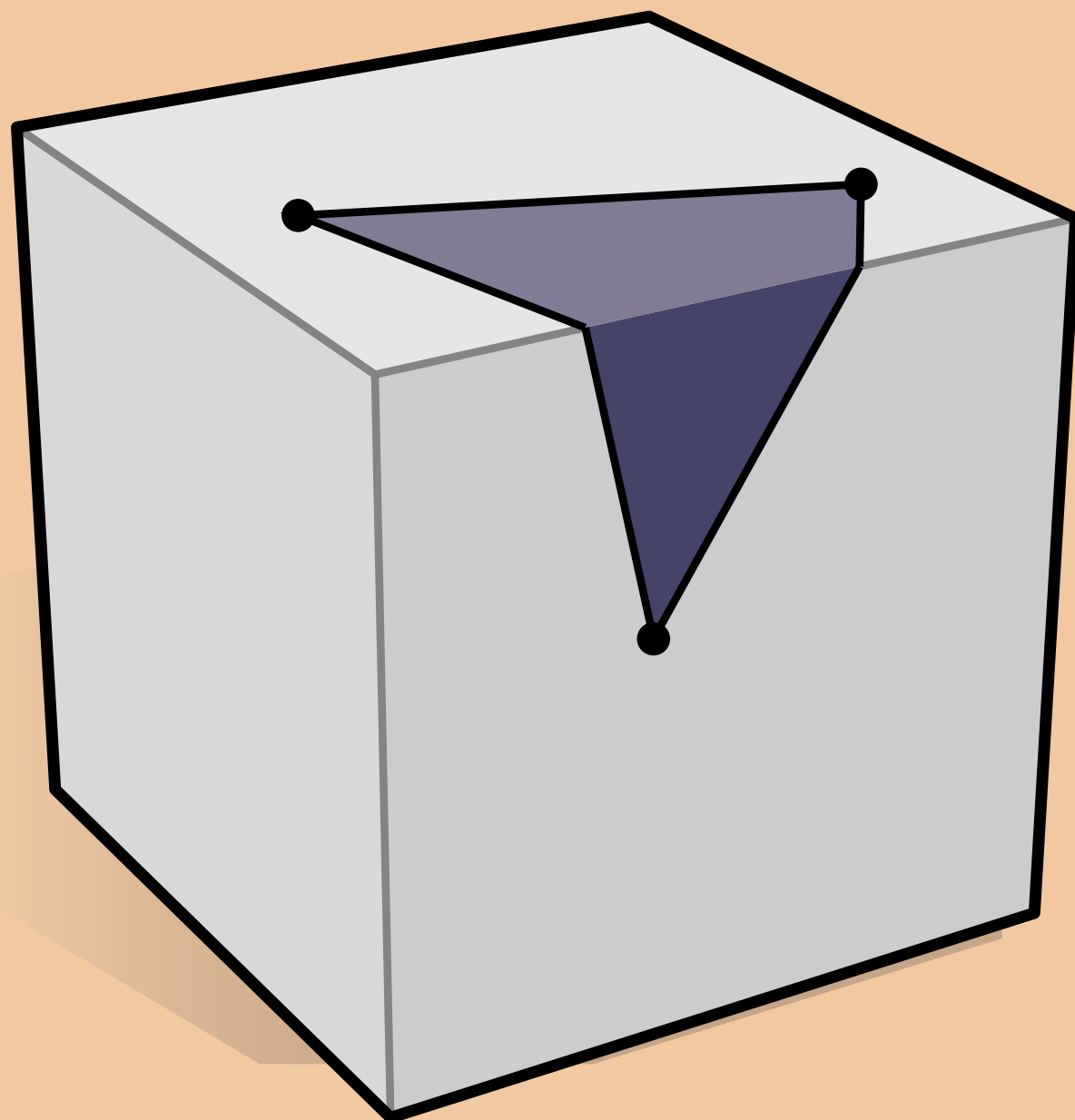
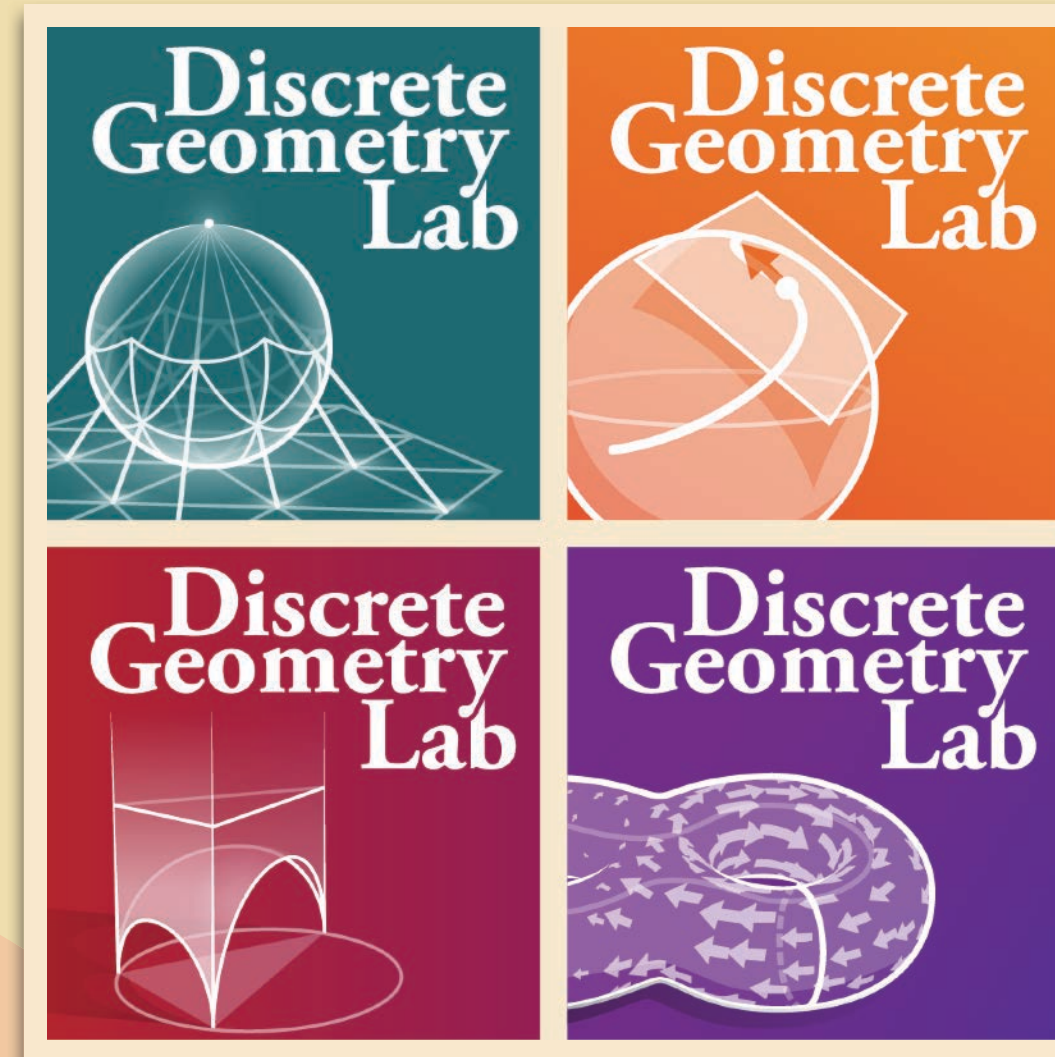
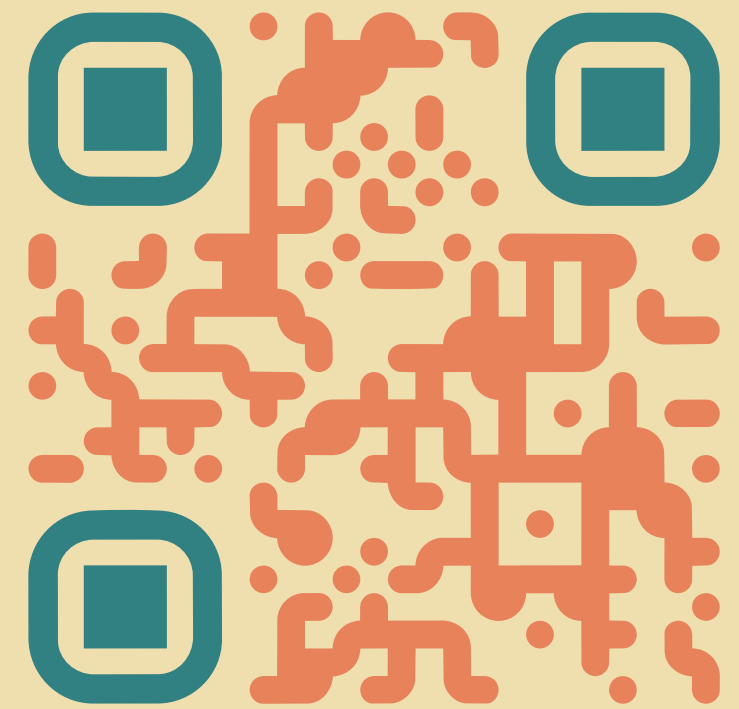


<https://github.com/MarkGillespie/intrinsic-triangulations-demo>

Thanks for listening

and consider signing up for my course (CS 5957/6957)
on *discrete differential geometry* this spring!

✉ mark.gillespie@utah.edu
🌐 geometry.cs.utah.edu



Supplemental Slides

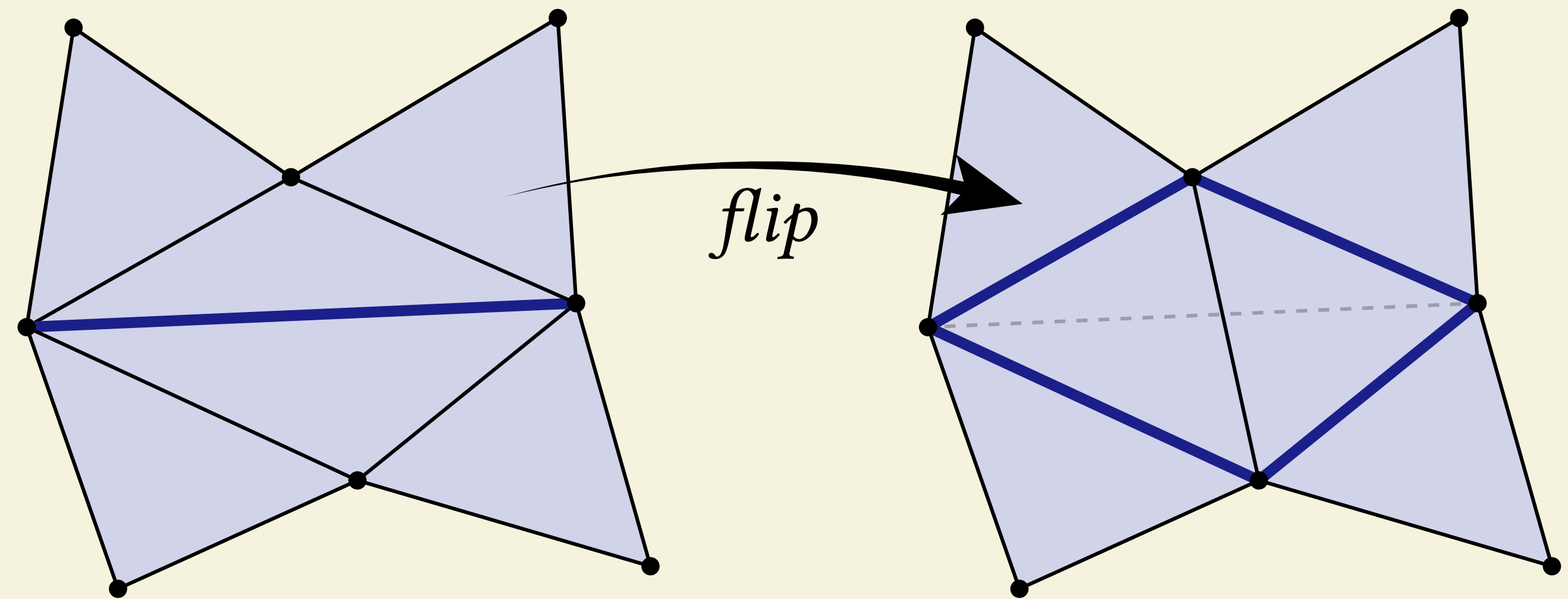
Lawson's Flip Algorithm

[Lawson 1972, 1977]

enqueue all edges

while queue is not empty:

- pop an edge
- flip it if it's not Delaunay
- if we flipped, enqueue neighbors not already in queue



Flip Algorithm Termination

[Bobenko & Springborn 2007]

For a Euclidean triangle t with sides a, b, c and area A , Musin [16] defines the *harmonic index* as

$$hrm(t) = \frac{a^2 + b^2 + c^2}{A}.$$

The harmonic index of a triangulation T with face set F is the sum of the harmonic indices of all triangles:

$$hrm(T) = \sum_{t \in F} hrm(t).$$

Proposition 14 *The harmonic index is proper.*

Proof In a PF surface, there may be infinitely many geodesic lines connecting two points, but for any $L \in \mathbb{R}$ only a finite number of them have length $\leq L$ [11]. Hence there are only a finite number of triangulations of a PF surface with marked points such that all edges are not longer than L . Now the Proposition follows from the inequality

$$hrm(T) \geq \frac{l_{\max}(T)}{A_{\text{tot}}},$$

where $l_{\max}(T)$ is the largest length of an edge of T and A_{tot} is the total area of the PF surface. Indeed, if $hrm(T) \leq M$, then $l_{\max}(T) \leq M A_{\text{tot}}$, and there are only finitely many triangulations satisfying this bound on edge lengths. $\square$

The following theorem was stated by Musin without proof [16].

Theorem 15 *With the notation and under the conditions of Rippa's Theorem*

$$hrm(T_1) \geq hrm(T_2)$$

and equality holds only if e is a Delaunay edge in T_1 as well. This implies that the harmonic index is minimal for a Delaunay triangulation T_D (and hence for all of them):

$$\min_T hrm(T) = hrm(T_D).$$

Proof The harmonic index of a triangle is

$$hrm(t) = 4(\cot \alpha + \cot \beta + \cot \gamma),$$

where α, β, γ are the angles of the triangle. (Because the area is $A = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, where h_a, h_b, h_c are the heights of the triangle; and $a/h_a = \cot \beta + \cot \gamma$, etc.)

For a triangulation T of (S, d) with vertex set V and $x \in V$ let $\delta_{x,T} : S \rightarrow \mathbb{R}$ be the function that is linear on the triangles of T , equal to 1 at x , and equal to 0 at all other marked points in V . Then

$$\sum_{x \in V} E(\delta_{x,T}) = \frac{1}{2} \sum_{\text{angles } \alpha \text{ in } T} \cot \alpha = \frac{1}{8} hrm(T).$$

Hence the theorem follows from Rippa's Theorem. $\square$

Status quo: remeshing

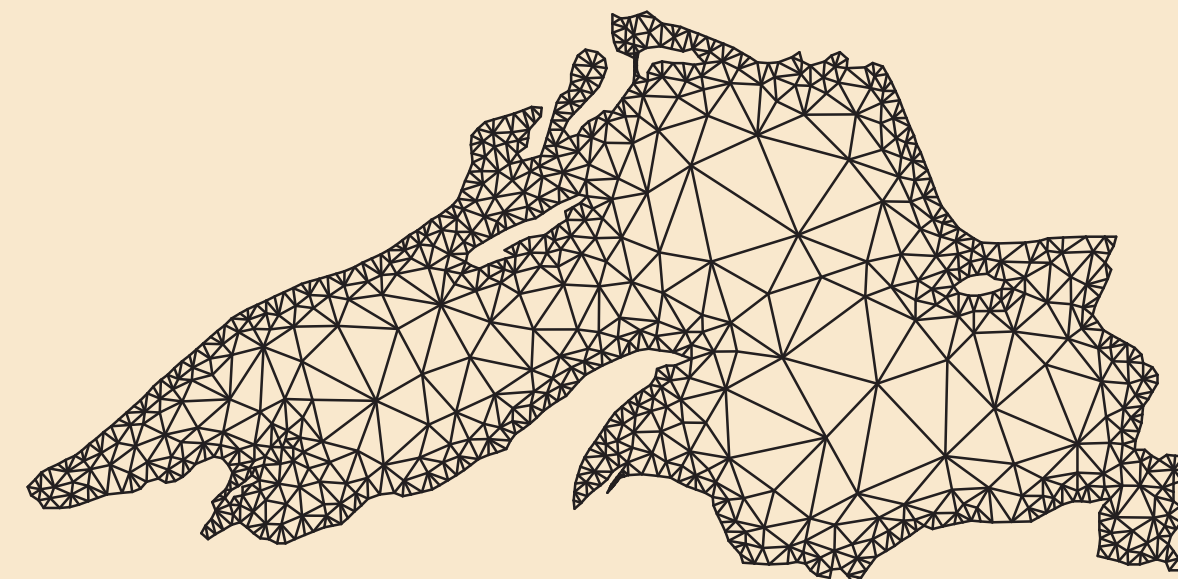
- State-of-the-art is robust but slow
- Techniques work volumetrically

Meshing is much easier in 2D

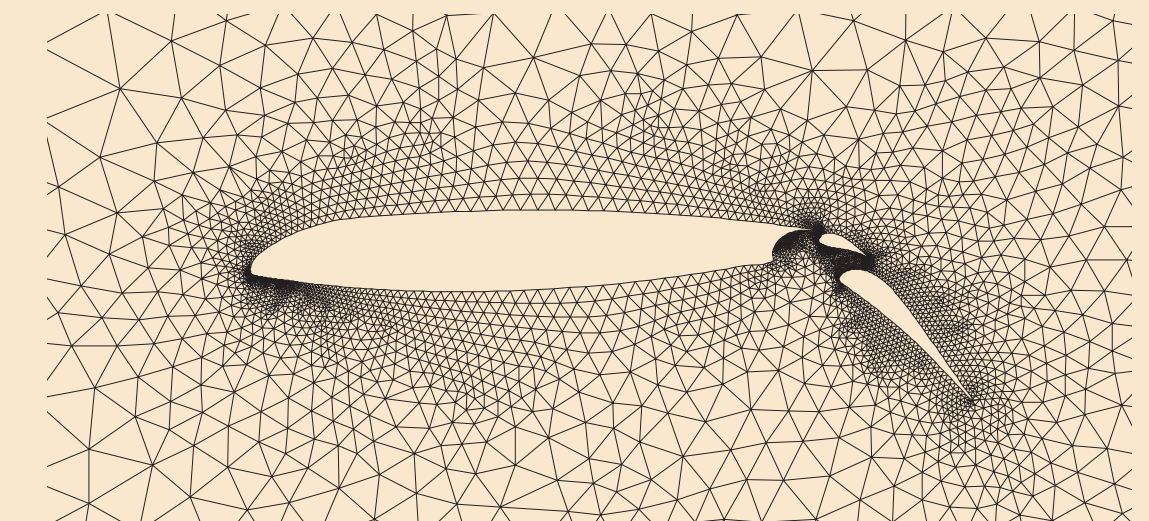
runtime:
47 minutes

has theoretical guarantees!

Generate high-quality meshes in milliseconds via Delaunay triangulations or Delaunay refinement [Chew 1993; Shewchuk 1997]



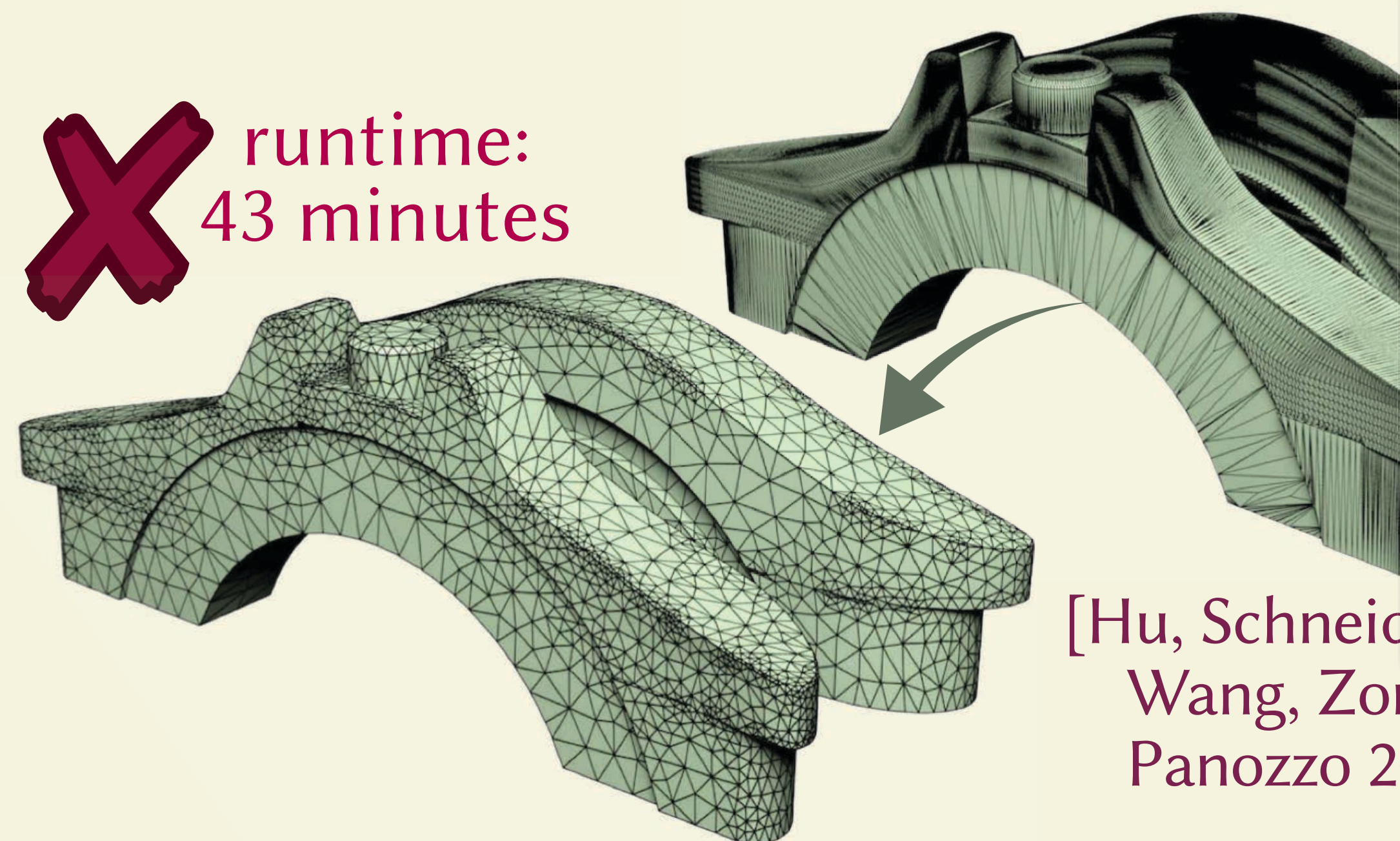
⌚ 80 milliseconds



⌚ 70 milliseconds

[Shewchuk 1997]

[Hu, Schneider
Wang, Zorin &
Panozzo 2020]



runtime:
43 minutes

Trade offs of extrinsic remeshing



3k faces



330k faces



3k faces

triangle quality



mesh size



geometric fidelity



Intrinsic triangulations sidestep the trade off

runs in
milliseconds

